SPPL: Probabilistic Programming with Fast Exact Symbolic Inference

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Abstract

We present the Sum-Product Probabilistic Language (Sppl), a new probabilistic programming language that automatically delivers exact solutions to a broad range of probabilistic inference queries. Sppl translates probabilistic programs into sum-product expressions, a new symbolic representation and associated semantic domain that extends standard sum-product networks to support mixed-type distributions, numeric transformations, logical formulas, and pointwise and set-valued constraints. We formalize Sppl via a novel translation strategy from probabilistic programs to sum-product expressions and give sound exact algorithms for conditioning on and computing probabilities of events. Sppl imposes a collection of restrictions on probabilistic programs to ensure they can be translated into sum-product expressions, which allow the system to leverage new techniques for improving the scalability of translation and inference by automatically exploiting probabilistic structure. We implement a prototype of Sppl with a modular architecture and evaluate it on benchmarks the system targets, showing that it obtains up to 3500x speedups over state-of-the-art symbolic systems on tasks such as verifying the fairness of decision tree classifiers, smoothing hidden Markov models, conditioning transformed random variables, and computing rare event probabilities.


Keywords: probabilistic programming, symbolic execution

1 Introduction

Reasoning under uncertainty is a well-established theme across diverse fields including robotics, cognitive science, natural language processing, algorithmic fairness, amongst many others [14, 21, 31, 60]. A common approach for modeling uncertainty is to use probabilistic programming languages (PPLs [29]) to both represent complex probability distributions and perform probabilistic inference within the language. There is growing recognition of the utility of PPLs for solving challenging tasks that involve probabilistic reasoning in various application domains [8, 27, 34, 35].

Probabilistic inference is central to reasoning about uncertainty and is a central concern for both PPL implementors and users. Several PPLs leverage approximate inference techniques [28, 59, 67], which have been used effectively in a variety of settings [11, 17, 55]. Drawbacks of approximate inference, however, include a lack of accuracy and/or soundness guarantees [18, 38]; difficulties with programs that combine continuous, discrete, or mixed-type distributions [11, 68]; challenges assessing the quality of iterative solvers [9]; and the substantial expertise needed to write custom inference programs that deliver acceptable performance [17, 39]. To address the shortcomings of approximate inference, several PPLs instead use exact symbolic techniques [6, 10, 23, 43, 69]. These languages can typically express a large class of models, using general computer algebra to solve queries. However, the generality of the symbolic computations causes them sometimes fail, even on problems with tractable solutions.

Our Work We introduce the Sum-Product Probabilistic Language (Sppl), a system that occupies a new point in the expressiveness vs. performance trade-off space for exact symbolic inference. A key idea in Sppl is to incorporate certain modeling restrictions that avoid the need for general computer algebra, instead using a new, specialized class of “sum-product” symbolic expressions to exactly represent probability distributions specified by Sppl programs. These new symbolic expressions extend and generalize sum-product networks [47], which are computational graphs that have received widespread attention for their clear probabilistic semantics and tractable properties for exact inference—see [64] for a comprehensive and curated literature review. These sum-product expressions are used to automatically obtain exact solutions to probabilistic inference queries about Sppl programs, which are fast and scalable in tractable regimes.
System Overview

Fig. 1 shows an overview of our approach. Given a probabilistic program written in Sppl (Lst. 2) a translator (Lst. 3) produces a sum-product expression that represents the prior distribution over all program variables. Given this expression and a query specified by the user, the Sppl inference engine returns an exact answer, where:

1. \texttt{simulate(Vars)} returns random samples of program variables from their joint probability distribution;
2. \texttt{prob(Event)} returns the probability of an event, which is a predicate on program variables;
3. \texttt{condition(Event)} returns a new sum-product expression for the posterior distribution over program variables, given that the specified event is true.

A key aspect of the system design in Fig. 1 is modularity: modeling, conditioning, and querying are factored into distinct stages that reflect the essential components of a Bayesian workflow. Moreover, the dashed back-edge in Fig. 1 indicates that the new sum-product expression returned by \texttt{condition} can be reused to interactively invoke additional queries on the posterior distribution. This closure property enables substantial runtime gains across multiple datasets and queries.

Trade-offs

Sppl imposes restrictions on probabilistic programs that specifically rule out the following constructs: (i) unbounded loops; (ii) multivariate numeric transformations; and (iii) arbitrary prior distributions on continuous parameters. As a result, Sppl is not designed to express model classes such as regression with a prior on real coefficients; neural networks; support-vector machines; spatial Poisson processes; urn processes; and hidden Markov models with unknown transition matrices. The aforesaid model classes cannot be represented as sum-product expressions, and most of them do not have tractable algorithms for exact inference.

We impose these restrictions to ensure that valid Sppl programs can always be translated into finite sum-product expressions, as opposed to general symbolic algebra expressions. The resulting sum-product expressions delivered by Sppl have a number of characteristics that make them a particularly useful translation target for probabilistic programs:

- Completeness and Decomposability: By satisfying important completeness and decomposability conditions from the literature [47, Defs. 4,5], sum-product expressions are guaranteed to represent normalized probability distributions.
- Efficient Factorization: By specifying multivariate probability distributions compositionally in terms of sums and products of simpler distributions, sum-product expressions can be simplified by algebraic “factorization” (Fig. 3d, Fig. 6a).
- Efficient Deduplication: When an Sppl program specifies a generative model with conditional independence structure, the translated sum-product expression typically contains identical subexpressions that can be “deduplicated” into a single logical node in memory (Fig. 3d, Fig. 6b).
- Efficient Caching: Inference algorithms for sum-product expressions proceed from root to leaves to root, allowing intermediate results to be cached and reused at deduplicated internal subexpressions in a depth-first graph traversal.
- Closure Under Conditioning: Sum-product expressions are closed under probabilistic conditioning (Thm. 4.1), which allows them to be reused across multiple datasets and inference queries about the same probabilistic program.
- Linear-Time Exact Inferences: For a well-defined class of common queries, inference scales linearly in the expression size (Thm. 4.3); when Sppl delivers a “small” expression after factorization and deduplication, inference is also fast.

It is well-known that a very large class of tractable models can be cast as sum-product networks [47, Thm. 2]. Sppl automatically constructs these representations from generative probabilistic programs that use standard constructs such as arrays, if/else branches, for-loops, and numeric and logical operators. To enable this translation, Sppl introduces new sum-product expressions and inference algorithms that extend standard sum-product networks by supporting (many-to-one) univariate transformations, mixed-type base measures, and pointwise and set-valued constraints. These constructs make Sppl expressive enough to solve prominent inference tasks in the PPL literature [2, 36, 46, 68] for which standard sum-product networks have not been previously
used. Example model classes include most finite discrete models, latent variable models with discrete hidden states and arbitrary observed states, and decision trees over discrete and continuous variables. Taken together, these characteristics make Sppl particularly effective for fast and scalable inference on tractable problems, with low variance runtime and complete, usable answers to users. Our experimental evaluation (Sec. 6) indicates that Sppl delivers these benefits on the problems it is designed to solve, whereas more general and expressive techniques in previous solvers [2, 4, 23] typically exhibit orders of magnitude worse performance on these problems, runtime has higher variance, and/or results may be unusable, i.e., with unsimplified symbolic integrals.

**Key contributions** We identify the following contributions:

- **New semantic domain for sum-product expressions** (Sec. 3) that extends sum-product networks [47] by including mixed-type distributions, numeric transforms, logical formulas, and events with pointwise and set-valued constraints.
- **Provably sound exact symbolic inference algorithms** (Sec. 4) based on a proof that sum-product expressions are closed under conditioning on any event that can be specified in the domain. We use these algorithms to build an efficient and multi-stage inference architecture that separates model translation, conditioning, and querying into distinct stages, enabling interactive workflows and computation reuse.
- **The Sum-Product Probabilistic Language** (Sec. 5), a PPL built on a novel translation semantics from generative code to sum-product expressions, which are used to deliver exact inferences to queries. We present optimization techniques to improve scalability of translation and inference by exploiting conditional independences and repeated structure.
- **Empirical measurements of efficacy** (Sec. 6) on inference tasks from the literature that Sppl targets, which show that it delivers substantial improvements over existing baselines, including up to 3500x speedup over state-of-the-art fairness verifiers [2, 4] and symbolic integration [23], as well as many orders of magnitude speedup over sampling-based inference [40] for computing rare event probabilities.

## 2 Overview

We next describe two examples that illustrate the programming style in Sppl and queries supported by the system.

### 2.1 Indian GPA Problem

The Indian GPA problem is a canonical example that has been widely considered in the probabilistic programming literature [44, 45, 48, 57, 68] for its use of a "mixed-type" random variable that takes both continuous and discrete values, depending on the random branch taken by the program.

**Specifying the Prior** Fig. 2a shows the generative process for three variables (Nationality, Perfect and GPA) of a student. The student’s nationality is either India or USA with equal probability (line 1). Students from India (line 2) have a 10% probability of a perfect 10 GPA (lines 3–4), otherwise the GPA is uniform over [0, 10] (line 5). Students from USA (line 6) have a 15% probability of a perfect 4 GPA (lines 6–7), otherwise the GPA is uniform over [0, 4] (line 8).

**Prior Sum-Product Expression** The graph in Fig. 2d shows the translated sum-product expression for this program, which represents a sampler for the distribution over program variables as follows: (i) if a node is a sum (+), visit a random child with probability equal to the weight of the edge pointing to the child; (ii) if a node is a product (×), visit each child exactly once, in no specific order; (iii) if a node is a leaf, sample a value from the distribution at the leaf and assign it to the variable at the leaf. Similarly, the graph encodes the joint distribution of the variables by treating (i) each sum node as a probabilistic mixture; (ii) each product node as a tuple of independent variables; and (iii) each leaf node as a primitive random variable. Thus, the prior distribution is:

\[
\Pr[\text{Nationality} = n, \text{Perfect} = p, \text{GPA} \leq g] = 0.5 \left[ \delta_{\text{India}}(n) \cdot (0.1 [\delta_{\text{True}}(p) \cdot 1 [10 \leq g]]) 
+ 0.9 \left[ (\delta_{\text{False}}(p) \cdot (g/10 \cdot 1 [0 \leq g < 10] + 1 [10 \leq g])) \right] \right] 
+ 0.5 \left[ (\delta_{\text{USA}}(n) \cdot (0.15 [\delta_{\text{True}}(p) \cdot 1 [4 \leq g]]) \right] 
+ 0.85 \left[ (\delta_{\text{False}}(p) \cdot (g/4 \cdot 1 [0 \leq g < 4] + 1 [4 \leq g])) \right].
\]

Fig. 2b shows Sppl queries for the prior marginal distributions of the three variables, plotted in Fig. 2c. The two jumps in the cumulative distribution function (CDF) of GPA at 4 and 10 correspond to the atoms that occur when Perfect is true. The piecewise linear behavior on [0, 4] and [4, 10] follows from the conditional uniform distributions of GPA.

**Conditioning the Program** Fig. 2f shows an example of the condition query, which specifies an event e on which to constrain executions of the program. An event is a predicate on (possibly transformed) program variables that can be used for both condition (Fig. 2f) and prob (Fig. 2c). Sppl is the first system with inference algorithms for sum-product expressions that handle predicates of this form. Given e, the object of inference is the full posterior distribution:

\[
\Pr[\text{Nationality} = n, \text{Perfect} = p, \text{GPA} \leq g \mid e] = \frac{\Pr[\text{Nationality} = n, \text{Perfect} = p, \text{GPA} \leq g, e]}{\Pr[e]}
\]

**Posterior Sum-Product Expression** Given the prior expression (Fig. 2d) and conditioning event e (Fig. 2f), Sppl produces a new expression (Fig. 2g) that specifies a distribution which is precisely equal to Eq. (2). From Thm. 4.1, conditioning an Sppl program on any event that can be specified in the language results in a posterior distribution that also admits an exact sum-product expression. Conditioning on e performs several transformations on the prior expression, which are:

1 For a real-valued random variable X, the cumulative distribution function \( F : \text{Real} \rightarrow [0, 1] \) is given by \( F(r) = \Pr[X \leq r] \).
India

PLDI ’21, June 20–25, 2021, Virtual, Canada

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1 Nationality ~ choice(’India’: 0.5, ’USA’: 0.5)
2 if (Nationality == ’India’):
3   Perfect ~ bernoulli(p=0.10)
4   if Perfect: GPA ~ atom(10)
5 else: GPA ~ uniform(0, 10)
6 else: # Nationality is ’USA’
7   Perfect ~ bernoulli(p=0.15)
8 if Perfect: GPA ~ atom(4)
9 else: GPA ~ uniform(0, 4)

(a) Probabilistic Program

(b) Example Queries on Marginal Probabilities

prob (Nationality == ’USA’);
prob (Perfect == 1);
prob (GPA <= x/10) # for x = 0, ..., 120

Conditioning the Program

(f) Conditioning the Program

5. Reweighting the branch probabilities at the root from
   [.5, .5] to [.33, .67] (same rules as in the previous item).

Fig. 2g shows the posterior expression obtained by applying
these transformations. Using this expression, the right-hand side of Eq. (2), which is the object of inference, is

\[
\Pr[\text{Nationality} = n, \text{Perfect} = p, \text{GPA} \leq g | e] = \Pr[\text{Perfect} = 1] 
\]
(Floats are shown to two decimal places.) We can now run the `prob` queries in Fig. 2b on the conditioned program to plot the posterior marginal distributions, which are shown in Fig. 2h. The example in Fig. 2 illustrates a typical modular workflow in Sppl (Fig. 1), where modeling (Fig. 2a), conditioning (Fig. 2f) and querying (Figs. 2b–2c) are separated into distinct and reusable stages that together express the main components of Bayesian modeling and inference.

### 2.2 Scalable Inference in a Hierarchical HMM

The next example shows how to perform efficient smoothing in a hierarchical hidden Markov model (HMM [42]) and illustrates the optimization techniques used by the Sppl translator (Sec. 5.1), which exploit conditional independence to ensure that the size of the sum-product expression grows linearly in the number of timesteps. The code box in Fig. 3a shows a hierarchical hidden Markov model with Bernoulli hidden states $Z_t$ and Normal–Poisson observations $(X_t, Y_t)$. The separated variable indicates whether the mean values of $X_t$ and $Y_t$ at $Z_t = 0$ and $Z_t = 1$ are well-separated. The $p$-transition vector specifies that the current state $Z_t$ switches from the previous state $Z_{t-1}$ with $20\%$ probability. This example leverages the Sppl `array`, `for`, and `switch-cases` statements, where the latter is a macro that expands to `if-else` statements (as in, e.g., the C language):

```plaintext
switch x cases (x'in values) {C}
  desugar ▼ if (x in values[0]) {C[x'/values[0]]} 
    elif...
    elif (x in values[n-1]) {C[x'/values[n-1]]},
```

where $n$ is the length of `values` and $C[x'/E]$ indicates syntactic replacement of $x$ with expression $E$ in command $C$.

The top and middle plots in Fig. 3b show a realization of $X$ and $Y$ that result from simulating the process for 100 time steps. The blue and orange regions along the $x$-axes indicate whether the true hidden state $Z$ is 0 or 1, respectively (these “ground-truth” values of $Z$ are not observed but need to be inferred from $X$ and $Y$). The bottom plot in Fig. 3b shows the exact posterior marginal probabilities $\Pr[Z_t = 1 \mid x_0, 99, y_0, 99]$ for each $t = 0, \ldots, 99$ as inferred by Sppl (an inference known as “smoothing”). These probabilities track the true hidden state, i.e., the posterior probabilities that $Z_t = 1$ are low in the blue and high in the orange regions.

Fig. 3c shows a “naive” sum-product expression for the distribution of all program variables up to the first two time steps. This expression is a sum-of-products, where the products in the second level are an enumeration of all possible realizations of program variables, so that the number of terms scales exponentially in the number of time steps. Fig. 3d shows the expression constructed by Sppl, which is (conceptually) based on factoring and sharing common terms in the two level sum-of-products in Fig. 3c. These factorizations and deduplications exploit conditional independences and repeated structure in the program (Sec. 5.1), which here delivers a expression whose size scales linearly in the number of time points. Sppl can also efficiently solve variants of smoothing, e.g., computing posterior marginals $\Pr[Z_t \mid x_{0:t}, y_{0:t}]$ (filtering) or the posterior joint $\Pr[Z_{0:t} \mid x_{0:t}, y_{0:t}]$ for any $t$.

### 3 A Core Calculus for Sum-Product Expressions

This section presents a semantic domain of sum-product expressions that generalizes sum-product networks [47] and enables precise reasoning about them. This domain will be used to (i) establish the closure of sum-product expressions under conditioning on events expressible in the calculus (Thm. 4.1); (ii) describe sound algorithms for exact Bayesian inference in our system (Appx. D); and (iii) describe a procedure for translating a probabilistic program into a sum-product expression in the core language (Sec. 5). Lst. 1 shows denotations of the key syntactic elements (Lst. 9 in Appx. A) in the calculus, which includes real and nominal outcomes (Lst. 1a); real transforms (Lst. 1b); predicates with pointwise and set-valued constraints (Lst. 1c); primitive distributions (Lst. 1e); and multivariate distributions specified compositionally as sums and products of primitive distributions (Lst. 1f).

#### Basic Outcomes

Random variables in the calculus take values in the Outcome := Real + String domain. The symbol × here indicates a sum (disjoint-union) data type, whose elements are formed by the injection operation, e.g., ↓Real Outcome for $r \in$ Real. This domain is used to model mixed-type random variables, such as $X$ in the following Sppl program:

```plaintext
X ~ normal(0, 1)
if (2 <= 0): X ~ "negative" # string
elif (0 < Z < 4): X ~ 2*exp(Z) # continuous real
elif (4 <= Z): X ~ atomic(4) # discrete real
```

The Outcomes domain denotes a subset of Outcome as defined by the valuation function $V$ (Lst. 1a). For example, $((b_1 r_1) (r_2 b_2))$ specifies a (open, closed, or clopen) real interval and $(s_1 \ldots s_n)^b$ is a set of strings, where $b = \#t$ indicates the complement (meta-variables such as $m$ indicate an arbitrary but finite number of repetitions of a domain variable or subexpression). The operations `union`, `intersection`, and `complement` operate on Outcomes in the usual way (while preserving certain semantic invariants, see Appx. B).

#### A Sigma Algebra of Outcomes

To speak precisely about random variables and measures on Outcome, we define a sigma-algebra $\mathcal{B}(\text{Outcome}) \subseteq \mathcal{P}(\text{Outcome})$ as follows:

1. Let $\tau_{\text{Real}}$ be the usual topology on Real.
2. Let $\tau_{\text{String}}$ be the discrete topology on String.
3. Let $\tau_{\text{Outcome}} := \tau_{\text{Real}} \uplus \tau_{\text{String}}$ be the disjoint-union topology on Outcome, where $U$ is open iff $(r \mid \downarrow \text{Real Outcome } r) \in U$ is open in Real and $(s \mid \downarrow \text{String } s) \in U$ is open in String.
4. Let $\mathcal{B}(\text{Outcome})$ be the Borel sigma-algebra of $\tau_{\text{Outcome}}$. 

Remark 3.1. As measures on Real are defined by their values on open intervals and measures on String on singletons, we can speak of probability measures on $\mathcal{B}(\text{Outcome})$ as mappings from Outcomes to $[0, 1]$. 

Real Transformations Lst. 1b shows real transformations that can be applied to variables in the calculus. The Identity Transform, written $\text{Id}(x)$, is a terminal subexpression of any Transform $t$ and contains a single variable name that specifies the "dimension" over which $t$ is defined. The list of all transforms is in Appx. C.1. The key operation involving transforms is computing the preimage of Outcomes $\nu$ under $t$ using $\text{preimg} : \text{Transform} \rightarrow \text{Outcomes} \rightarrow \text{Outcomes}$ which satisfies the following properties:

$$
(\downarrow \text{Real Outcome} r) \in \mathcal{V} \left[ \text{preimg} \ t \nu \right] \iff T \left[ r \right] (r) \in \mathcal{V} \left[ \nu \right]
$$

$$
(\downarrow \text{String Outcome} s) \in \mathcal{V} \left[ \text{preimg} \ t \nu \right] \iff (t \in \text{Identity}) \land (s \in \mathcal{V} \left[ \nu \right]).
$$

Appx. C.2 presents a symbolic solver that implements $\text{preimg}$ for each Transform, which is leveraged to enable exact probabilistic inferences on transformed variables in Sppl. Fig. 4 and Appx. C.3 show example inferences with transforms.
\( \textbf{Events} \) Lst. 1c shows the Event domain, which specifies predicates on variables. The valuation \( \mathbb{E} [\cdot] : \text{Var} \rightarrow \text{Outcomes} \) of an Event takes a variable and returns the set \( \mathcal{U} \in \text{Outcomes} \) of elements that satisfy the predicate along dimension \( x \), leveraging the properties of \textit{preimg}. This domain specifies measurable sets of \( n \)-dimensional variables \( x_1, \ldots, x_n \) as follows: let \( \sigma_{\text{gen}}(A_1, A_2, \ldots) \) be the sigma-algebra generated by \( A_1, A_2, \ldots \), and define \( \mathcal{B}^n(\text{Outcome}) := \sigma_{\text{gen}}(\prod_{1 \leq i \leq n} U_i \mid \forall 1 \leq i \leq n, U_i \in \mathcal{B}(\text{Outcomes})) \). In other words, \( \mathcal{B}^n(\text{Outcome}) \) is the \( n \)-fold product sigma-algebra generated by open rectangles of Any. \( e \in \text{Event} \) specifies a measurable set \( U \in \mathcal{B}^n(\text{Outcome}) \), whose \( i \)-th coordinate \( U_i = \mathbb{E}[x_i] \) if \( x_i \) \in \text{vars \( e \)} \), and \( U_i = \text{Outcomes} \) otherwise. Transform in \( e \) is solved and any \text{Var} that does not appear in \( e \) is marginalized, as in the next example.
Figure 4. Inference on a stochastic many-to-one transformation of a real random variable in SpPPL.

real (on the integers) and nominal distributions. The notation $D[\sigma]$ of a Distribution is a distribution on Outcomes (recall Remark 3.1). For example, $\text{distPr}(F, r_1, r_2)$ is the restriction of $F$ to a positive measure interval $[r_1, r_2]$. The distributions specified by $\text{distPr}$ and $\text{distT}$ can be simulated using a variant of the integral probability transform (Prop. A.1 in Appx. A), which also defines their sampling semantics.

**Sum-Product Expressions** Lsts. 1d and 1f show the probability density and distribution semantics of the SPE domain, respectively, whose elements are probability measures. The following conditions specify well-definedness for SPE:

(C1) $\forall \text{Leaf}(x \ d \ \sigma). \ x \in \sigma$ and $\sigma(x) = \text{Id}(x)$.
(C2) $\forall \text{Leaf}(x \ d \ \sigma). \ \text{dom}(\sigma) = \{x, x_1, \ldots, x_m\}$ for some $m > 0$ then $\forall 1 \leq i \leq m. (\text{vars}(\sigma(x_i))) \subset \{x, x_1, \ldots, x_{i-1}\}$.
(C3) $\forall (S_1 \odot \cdots \odot S_m). \ \text{Observe}(\sigma \cap \{S_l\}) \neq \emptyset$.
(C4) $\forall (S_1 \odot w_1) \odot \cdots \odot (S_m \odot w_m). \ \forall i. (\text{scope}(S_i) = (\text{scope}) \cap \sigma)$.
(C5) $\forall (S_1 \odot w_1) \odot \cdots \odot (S_m \odot w_m). \ w_1 + \cdots + w_m > 0$.

For Leaf, (C1) ensures that $\sigma$ maps the leaf variable $x$ to the Identity Transform and (C2) ensures there are no cyclic dependencies or undefined variables in Environment $\sigma$. Condition (C3) ensures the scopes of all children of a Product are disjoint and (C4) ensures the scopes of all children of a Sum are identical, which together ensure completeness and decomposability from sum-product networks [47, Defs. 4, 5].

In Lst. 1f, the denotation $\text{P}[S]$ of $S \in$ SPE is a map from $e \in$ Event to its probability under the $n$-dimensional distribution defined by $S$, where $n := |\text{scope}(S)|$ is the number of variables in $S$. A terminal node $\text{Leaf}(x \ d \ \sigma)$ is comprised of a Var $x$, Distribution $d$, and Environment $\sigma$ that maps other variables to a Transform of $x$, e.g., $Z \mapsto \text{Poly}(\text{Root}(\text{Id}(X) \ 2))$ [11, 5].

When assessing the probability of $e$ at a Leaf, $\text{subenv}$ (Lst. 13 in Appx. A) rewrites $e$ as an Event $e'$ on one variable $x$, so that the probability of Outcomes that satisfy $e$ is exactly $D[\sigma][x \mapsto (x \in e)]$.

$e$ is a weighted average of the probabilities under each subexpression. For a Product, the semantics are defined in terms of $\langle \text{dnf} e \rangle$ (Lst. 15 in Appx. B), leveraging inclusion-exclusion. In Lst. 1d, the denotation $\text{P}_0[S]$ defines the density semantics of SPE, used for measure zero events such as $X = 3, Y = \pi, Z = \text{“foo”}$ under a mixed-type base measure. These semantics, which define the density as a pair, adapt "lexicographic likelihood-weighting", an approximate inference algorithm for discrete-continuous Bayes Nets [68], to exact inference using "lexicographic enumeration" for SPE.

## 4 Conditioning Sum-Product Expressions

We next present the main theoretical result for exact inference on probability distributions defined by an expression $S \in$ SPE and describe the inference algorithm for conditioning on an Event (Lst. 1c) in the core calculus, which includes transformations and predicates with set-valued constraints.

**Theorem 4.1 (Closure under conditioning).** Let $S \in$ SPE and $e \in$ Event be given, where $\text{P}[S] > 0$. There exists an algorithm which, given $S$ and $e$, returns $S' \in$ SPE such that, for all $e' \in$ Event, the probability of $e'$ according to $S'$ is equal to the conditional probability of $e'$ given $e$ according to $S$, i.e.,

$$\text{P}[S'](e') \equiv \text{P}[S](e' | e) \equiv \frac{\text{P}[S](e' \cap e)}{\text{P}[S](e)} \quad (5)$$

Thm. 4.1 is a structural conjugacy property [20] for the family of probability distributions defined by the SPE domain, where both the prior and posterior are identified by elements of SPE. We establish Eq. (5) constructively, by describing a new algorithm condition : SPE $\rightarrow$ Event $\rightarrow$ SPE that satisfies

$$\text{P}(\text{condition } S \ e) \ (e') = \text{P}[S](e' | e) \quad (6)$$

for all $e, e' \in$ Event with $\text{P}[S] > 0$. Refer to Appx. D for the proof. Fig. 5 shows a conceptual example of how
We next present a probabilistic language called $S$. As with sum-product networks, $S$ is also closed under
conditioning on a Conjunction of possibly measure zero
equality constraints on non-transformed variables, which
appear in many PPL interfaces [17, 41, 51]. Appx. D3 presents
the $\text{condition}_{\text{SPPL}}$ algorithm for inference on such
events, leveraging the generalized mixed-type density semantics in Lst. 1d.

The next result, Thm. 4.3, states a sufficient requirement
for inference using (condition $S$) to scale linearly in the size
of $S$, which holds for both zero and positive measure events.

**Theorem 4.3.** The runtime of (condition $S$) scales linearly
in the number of nodes in the graph representing $S$ whenever $e$
is a single Conjunction $(t_1 \in v_1) \land \cdots \land (t_m \in v_m)$
of Containment constraints on non-transformed variables.

## 5 Translating Probabilistic Programs to
Sum-Product Expressions

We next present a probabilistic language called Sppl and
show how to translate each program in the language to an
element $S \in \text{SPE}$ that symbolically represents (via $P[S]$)
the probability distribution specified by the program. As in Fig. 1,
$S$ can then be used to answer queries about an Event $e$:

**simulate**: Samples from the distribution defined by $P[S]$;

**prob**: Computes the probability of $e$, using $P[S]$ $e$ (Lst. 1f);

**condition**: Conditions on $e$, using condition (Eq. (6)).

Lst. 2 shows the source syntax of Sppl, which contains standard
constructs of an imperative language such as $\text{array}$
data structures, if-else statements, and bounded for loops.

The switch-case macro is defined in Eq. (4). Random variables are defined using “sample” ($) and condition($E$)
can be used to restrict executions to those for which $E$ $\text{Expr}$
evaluates to $\#t$ as part of the prior definition. Lst. 3
defines a relation $(C, S) \rightarrow_{\text{SPE}} S'$, which translates a “current”
$S \in \text{SPE}$ and $C$ in Command into $S' \in \text{SPE}$, where the initial

![Disjoined Region](https://via.placeholder.com/150)

**Figure 5.** Conditioning a Product $S$ on an Event $e$ that
defines a union of hyperrectangles in $\text{Real}$. The inference
algorithm partitions the region into a disjoint union, in this
case converting two overlapping regions into five disjoint
regions. The result is a Sum-of-Product, where each child is
the restriction of $S$ to one of the disjoint hyperrectangles.

**condition works**, where the prior distribution is a Product $S$
and the conditioned distribution is a Sum-of-Product $S'$. Fig. 4 shows an example of the closure property when the
sum-product expression has transformed random variables
(details in Appx. C).

**Remark 4.2.** Thm. 4.1 refers to a positive probability Event
$e$. As with sum-product networks, $S$ is also closed under
conditioning on a Conjunction of possibly measure zero
equality constraints on non-transformed variables, which
appear in many PPL interfaces [17, 41, 51]. Appx. D3 presents
the $\text{condition}_{\text{SPPL}}$ algorithm for inference on such
events, leveraging the generalized mixed-type density semantics in Lst. 1d.

The next result, Thm. 4.3, states a sufficient requirement
for inference using (condition $S$) to scale linearly in the size
of $S$, which holds for both zero and positive measure events.

**Theorem 4.3.** The runtime of (condition $S$) scales linearly
in the number of nodes in the graph representing $S$ whenever $e$
is a single Conjunction $(t_1 \in v_1) \land \cdots \land (t_m \in v_m)$
of Containment constraints on non-transformed variables.

## 5 Translating Probabilistic Programs to
Sum-Product Expressions

We next present a probabilistic language called Sppl and
show how to translate each program in the language to an
element $S \in \text{SPE}$ that symbolically represents (via $P[S]$)
the probability distribution specified by the program. As in Fig. 1,
$S$ can then be used to answer queries about an Event $e$:

**simulate**: Samples from the distribution defined by $P[S]$;

**prob**: Computes the probability of $e$, using $P[S]$ $e$ (Lst. 1f);

**condition**: Conditions on $e$, using condition (Eq. (6)).

Lst. 2 shows the source syntax of Sppl, which contains standard
constructs of an imperative language such as $\text{array}$
data structures, if-else statements, and bounded for loops.

The switch-case macro is defined in Eq. (4). Random variables are defined using “sample” ($) and condition($E$)
can be used to restrict executions to those for which $E$ $\text{Expr}$
evaluates to $\#t$ as part of the prior definition. Lst. 3
defines a relation $(C, S) \rightarrow_{\text{SPE}} S'$, which translates a “current”
$S \in \text{SPE}$ and $C$ in Command into $S' \in \text{SPE}$, where the initial

![Disjoined Region](https://via.placeholder.com/150)

**Figure 5.** Conditioning a Product $S$ on an Event $e$ that
defines a union of hyperrectangles in $\text{Real}$. The inference
algorithm partitions the region into a disjoint union, in this
case converting two overlapping regions into five disjoint
regions. The result is a Sum-of-Product, where each child is
the restriction of $S$ to one of the disjoint hyperrectangles.

**condition works**, where the prior distribution is a Product $S$
and the conditioned distribution is a Sum-of-Product $S'$. Fig. 4 shows an example of the closure property when the
sum-product expression has transformed random variables
(details in Appx. C).

**Remark 4.2.** Thm. 4.1 refers to a positive probability Event
$e$. As with sum-product networks, $S$ is also closed under
conditioning on a Conjunction of possibly measure zero
equality constraints on non-transformed variables, which
appear in many PPL interfaces [17, 41, 51]. Appx. D3 presents
the $\text{condition}_{\text{SPPL}}$ algorithm for inference on such
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in the number of nodes in the graph representing $S$ whenever $e$
is a single Conjunction $(t_1 \in v_1) \land \cdots \land (t_m \in v_m)$
of Containment constraints on non-transformed variables.
When they exist, to build compact sum-product expressions. As discrete Bayesian networks can be encoded as (Fig. 6a), provided that the new expression satisfies conditions (C1)–(C5). Factorization plays a key role in the (IfElse) rule of $\rightarrow_{\text{SPE}}$ since all statements before the \texttt{if} and \texttt{else} branches, we find empirically that previous statements that are probabilistically independent of statements inside the branches often produce identical subexpressions that can be factored out.

**Deduplication** When a sum-product expression contains duplicate subexpressions that cannot be factored out without violating the definedness conditions, we instead resolve duplicates into a single logical representation. Fig. 6b shows an example where the left and right components of the original expression contain an identical subexpression $S$ (in blue), but factorization would lead to an invalid sum-product expression. The optimizer represents the computation graph of this expression using a single data structure $S$ shared by the left and right subtrees (see also Figs. 3c–3d).

**Memoization** While deduplication reduces memory overhead, memoization is used to reduce runtime overhead. Consider either SPE in Fig. 6b: calling \texttt{condition} on the Sum root will dispatch the query to the left and right subexpressions (Lst. 6b). We cache the results of \texttt{condition $S$ e} or $P[S]e$ when $S$ is visited in the left subtree to avoid recomputing the result when $S$ is visited again in the right subtree via a depth-first traversal. Memoization delivers large runtime gains not only for solving queries but also for detecting duplicates returned by \texttt{condition} in the (IfElse) translation step.

**Measurements** Table 1 shows measurements of performance gains delivered by the factorization and deduplication optimizations on seven benchmarks. Compression ratios range between $1.2x$ to $1.64\times10^{13}x$ and are highest in the presence of independence or repeated structure. The deduplication and memoization optimizations together enable fast detection of duplicate subtrees by comparing logical memory addresses of internal nodes in $O(1)$ time, instead of computing hash functions that require an expensive subtree traversal.
6 Evaluation

We implemented a prototype of Sppl\(^2\) and evaluated its performance on benchmark problems from the literature. Sec. 6.1 compares the runtime of verifying fairness properties of decision trees using Sppl to FairSquare [2] and VeriFair [4], two state-of-the-art fairness verification tools. Sec. 6.2 compares the runtime of conditioning and querying probabilistic programs using Sppl to PSI [23], a state-of-the-art tool for exact symbolic inference. Sec. 6.3 compares the runtime of computing exact rare event probabilities in Sppl to sampling-based estimation in BLOG [40]. Experiments were run on Intel i7-8665U 1.9GHz CPU with 16GB RAM.

6.1 Fairness Benchmarks

Characterizing the fairness of classification algorithms is a growing application area in machine learning [21]. Recently, Albarghouthi et al. [2] precisely cast the problem of verifying the fairness of a classifier in terms of computing ratios of conditional probabilities in a probabilistic program that specifies the data generating and classification processes. Briefly, if (i) \( D \) is a decision program that classifies whether applicant \( A \) should be hired; (ii) \( H \) is a population program that generates random applicants; and (iii) \( \phi_m \) (resp. \( \phi_q \)) is a predicate on \( A \) that is true if the applicant is a minority (resp. qualified), then \( D \) is \( \epsilon \)-fair on \( H \) (where \( \epsilon > 0 \)) if

\[
\frac{\Pr_{A \sim H} [D(A) \mid \phi_m(A) \land \phi_q(A)]}{\Pr_{A \sim H} [D(A) \mid \neg \phi_m(A) \land \phi_q(A)]} > 1 - \epsilon,
\]

(i.e., the probability of hiring a qualified minority applicant is \( \epsilon \)-close to that of hiring a qualified non-minority applicant.

In this evaluation, we compare the runtime needed by Sppl to obtain a fairness judgment (Eq. (7)) for machine-learned decision and population programs against the FairSquare [2] and VeriFair [4] solvers. We evaluate performance on the decision tree benchmarks from Albarghouthi et al. [2, Sec. 6.1], which are one-third of the full benchmark set. Sppl cannot solve the neural network and support-vector machine benchmarks, as they contain multivariate transforms which do not have exact tractable solutions and are ruled out by the Sppl restriction (R3). FairSquare and VeriFair can express these benchmarks as they have approximate inference.

Table 2 shows the results. The first column shows the decision making program (\( D^n \) means "decision tree" with \( n \) conditionals); the second column shows the population model used to generate data; the third column shows the lines of code (in Sppl); and the fourth column shows the result of the fairness analysis (FairSquare, VeriFair, and Sppl produce the same judgment on all fifteen benchmarks). The remaining columns show the runtime and speedup factors. We note that Sppl, VeriFair, and FairSquare are all implemented in Python, which allows for a fair comparison. The measurements indicate that Sppl consistently obtains probability estimates in milliseconds, whereas the two baselines can each require over 100 seconds. The Sppl speedup factors are up to 3500x (vs. VeriFair) and 2934x (vs. FairSquare). We further observe that the runtimes in FairSquare and VeriFair vary significantly. For example, VeriFair uses rejection sampling to estimate Eq. (7) with a stopping rule to determine when the estimate is close enough, leading to unpredictable runtime (e.g., \( >100 \) seconds for \( D^n_4 \) but \( <1 \) second for \( D^n_4 \), Bayes Net. 2). FairSquare, which uses symbolic volume computation and hyperrectangle sampling to approximate Eq. (7), is faster than VeriFair in some cases (e.g., \( D^n_4 \)), but times out in others (\( D^n_4 \), Bayes Net. 2). In contrast, Sppl, computes exact probabilities for Eq. (7) and its runtime does not vary significantly across the various benchmark problems. The performance–expressiveness trade-off here is that Sppl computes exact probabilities and is substantially faster on the decision tree problems that it can express. FairSquare and VeriFair compute approximate probabilities that enable them to express more fairness problems, at the cost by of a higher and less predictable runtime on the decision trees.

<table>
<thead>
<tr>
<th>Decision Program</th>
<th>Population Model</th>
<th>Lines of Code</th>
<th>Fairness Judgment</th>
<th>Wall-Clock Runtime (seconds)</th>
<th>Sppl Speedup Factor vs. FairSquare vs. VeriFair</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Sppl</td>
<td>FairSquare</td>
</tr>
<tr>
<td>DT(_4)</td>
<td>Independent</td>
<td>15</td>
<td>Unfair</td>
<td>1.4</td>
<td>16.0</td>
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<tr>
<td>Bayes Net. 1</td>
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<td>Unfair</td>
<td>2.5</td>
<td>1.27</td>
<td>0.03</td>
</tr>
<tr>
<td>Bayes Net. 2</td>
<td>29</td>
<td>Unfair</td>
<td>6.2</td>
<td>0.91</td>
<td>0.03</td>
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<tr>
<td>DT(_4)</td>
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<td>Fair</td>
<td>2.7</td>
<td>105</td>
</tr>
<tr>
<td>Bayes Net. 1</td>
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<td>Fair</td>
<td>15.5</td>
<td>152</td>
<td>0.07</td>
</tr>
<tr>
<td>Bayes Net. 2</td>
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<td>Fair</td>
<td>70.1</td>
<td>151</td>
<td>0.08</td>
</tr>
<tr>
<td>DT(_4)</td>
<td>Independent</td>
<td>36</td>
<td>Fair</td>
<td>4.1</td>
<td>13.6</td>
</tr>
<tr>
<td>Bayes Net. 1</td>
<td>49</td>
<td>Unfair</td>
<td>12.3</td>
<td>1.58</td>
<td>0.08</td>
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<td>Unfair</td>
<td>30.3</td>
<td>2.02</td>
<td>0.08</td>
</tr>
<tr>
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<td>62</td>
<td>Fair</td>
<td>5.1</td>
<td>2.01</td>
</tr>
<tr>
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<td>21.6</td>
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<td>24.5</td>
<td>0.12</td>
</tr>
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<td>DT(_4)</td>
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<td>15.6</td>
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<td>140</td>
<td>0.1</td>
<td>0.01</td>
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</tbody>
</table>

\(^2\)Available in supplement and online at https://github.com/probcomp/sppl.
We next compare Sppl to PSI [23], a state-of-the-art symbolic inference engine, on benchmark problems that include discrete, continuous, and transformed random variables. PSI can express more inference problems than Sppl, as it uses general computer algebra without having restrictions (R3) and (R4) in Sppl. As a result, Sppl can solve 14/21 benchmarks listed in [23, Table 1]. We first discuss key architecture novelties in Sppl that contribute to its performance gains.

**Workflow Comparison** In Sppl, the multi-stage modeling and inference workflow (Fig. 7a) involves three steps that reflect the key elements of a Bayesian inference problem:
- (S1) Translating the model program into a prior SP S.
- (S2) Conditioning S on data to obtain a posterior SP S′.
- (S3) Querying S′, using, e.g., `prob` or `simulate`.

An advantage of this multi-stage workflow is that multiple tasks can be run at a given stage without rerunning previous stages. For example, multiple datasets can be observed in (S2) without translating the prior expression in (S1) once per dataset; and, similarly, multiple queries can be run in (S3) without conditioning on data in (S2) once per query. In contrast, PSI adopts a single-stage workflow (Fig. 7b), where a single program contains the prior distribution over variables, “observe” (i.e., “condition”) statements for conditioning on a dataset, and a “return” statement for the query. PSI converts the program into a symbolic expression for the distribution over the return value: if this expression is “complete” (i.e., no unevaluated symbolic integrals) it can be used to obtain interpretable answers (e.g., for plotting or tabulating); otherwise, the result is “partial” and is too complex to be used for practical purposes. A consequence of the single-stage workflow in a system like PSI is that the entire solution is recomputed from scratch on a per-dataset or per-query basis.

**Runtime Comparison** Table 4 compares the runtime of Sppl and PSI on seven benchmarks problems: Digit Recognition [23]; TrueSkill [36]; Clinical Trial [23]; Gamma transforms (described below); Student Interviews [36] (two variants); and Markov Switching (two variants, from Sec. 2.2); The second column shows the distributions in each benchmark, which include continuous, discrete, and transformed variables. The third column shows the number of datasets on which to condition the program. The three next columns show the time needed to translate the program (stage (S1)), condition the program on a dataset (stage (S2)), and query the posterior (stage (S3))—entries in the latter two columns are written as n × t, where n is the number of datasets and t the average time per dataset. In PSI, modeling and observing data are a single stage, shown in the merged gray cell. Querying the posterior is instantaneous when PSI produces a result without unsimplified integrals and times out when the result is unusable (∞). The final column shows the overall runtime needed to solve all tasks on the n datasets.

For benchmarks that both systems solve completely, Sppl realizes speedups between 3x (Digit Recognition) to 3600x (Markov Switching). In addition, the measurements show the advantage of our multi-stage workflow; for example, in TrueSkill, which uses a Poisson–Binomial distribution, Sppl translation (3.4 seconds) is more expensive than both conditioning on data (0.7 seconds) and querying (0.1 seconds), which highlights the benefit of amortizing the translation cost over several datasets or queries. In PSI, solving TrueSkill takes 2 × 41.6 seconds, but the solution contains unsimplified integrals and is thus unusable. The Markov Switching and Student Interviews benchmarks show that PSI may not perform well in the presence of many discrete random variables.

The Gamma Transform benchmark tests the robustness of many-to-one transforms of random variables (Lst. 1b), where X ∼ Gamma(3, 1); Y = 1/\exp X^2 if X < 1 and Y = 1/\ln X otherwise; and Z = − Y^3 + Y^2 + 6Y. Each of the n = 5 datasets specifies a different constraint φ(Z) and a query about the posterior Y | φ(Z), which needs to compute and integrate out X | φ(Z). PSI reports that there is an error in its answer for all five datasets, whereas Sppl, using the symbolic transform solver from Appx. C.2, solves all five problems effectively.

Table 3 compares the runtime variance using Sppl and PSI for four of the benchmarks in Table 4, repeating one query over 10 datasets. In all benchmarks, the Sppl variance is lower than that of PSI, with a maximum standard deviation σ = 0.5 sec. In contrast, the spread of PSI runtime is high.
for Student Interviews ($\sigma = 540$ sec, range $64\text{–}1890$ sec) and Clinical Trial ($\sigma = 153$ sec, range $2.75\text{–}470$ sec). In PSI, the symbolic analyses are sensitive to the numeric values in the dataset, leading to unpredictable runtime across different datasets, even for a fixed query pattern. In Sppl, the runtime depends only on the query pattern not the observed data and therefore behaves predictably across different datasets.

As with the fairness benchmarks in Sec. 6.1, PSI trades off expressiveness with efficacy on tractable problems, and our measurements show that its runtime and memory do not scale well or are unpredictable on benchmarks that Sppl solves very efficiently. Moreover, the evaluations show that PSI can return unusable inference results to the user and that it needs to recompute entire symbolic solutions from scratch for each new dataset or query, whereas Sppl is less expressive than PSI but carries neither of these limitations.

### 6.3 Comparison to Sampling-Based Estimates

We next compare the runtime and accuracy of estimating probabilities of rare events in a canonical Bayesian network [33] using Sppl and BLOG [40]. As discussed by Koller and Friedman [33, Sec 12.13], rare events are the rule, not the exception, in many applications, as the probability of a predicate $\phi(X)$ decreases exponentially with the number of observed variables in $X$. Small estimation errors can magnify substantially when, e.g., taking ratios of probabilities.

In Fig. 8, each subplot shows the runtime and probability estimates for a low-probability predicate $\phi$. In BLOG, the rejection sampler estimates the probability of $\phi$ by computing the fraction of times it holds in a size $n$ i.i.d. random sample from the prior. The horizontal red line shows the “ground truth” probability. The x marker shows the runtime needed by Sppl to (exactly) compute the probability and the dots show the estimates from BLOG with increasing runtime (i.e., more samples $n$). Sppl consistently returns an exact answer in less than 2ms. The accuracy of BLOG estimates improve as the runtime increases: by the strong law of large numbers, these estimates converge to the true value, but the fluctuations for any single run can be large (the standard error decays as $1/\sqrt{n}$). Each “jump” corresponds to a new sample $X^{j}$ that satisfies $\phi(X^{j})$, which increases the estimate. Without ground truth, it is hard to predict how much computation is needed for BLOG to obtain accurate results: estimates for predicates with $\log \phi = -12.73$ and $\log \phi = -17.32$ did not converge within the allotted time, while those for $\log \phi = -14.48$ converged after 180 seconds.

### 7 Related Work

Sppl is distinguished by being the first system to deliver exact symbolic inference by translating probabilistic programs to sum-product expressions, which extend and generalize sum-product networks. We briefly discuss related approaches.

**Symbolic Integration** Several systems deliver exact inference by translating a probabilistic program and observed dataset into a symbolic expression whose solution is the answer to the query [6, 10, 23, 43, 69]. Our approach to exact inference, which uses sum-product expressions instead of general computer algebra, enables effective performance on a range of models and queries, primarily at the expense of the expressiveness of the language on continuous priors. The state-of-the-art solver, PSI [23], can effectively solve many inference problems that Sppl cannot express due to restrictions (R1)–(R4), including higher-order programs [24]. However, comparisons on benchmarks that Sppl targets (Sec. 6.2) find PSI has less scalable and higher variance runtime, and can return partial results with unsimplified symbolic integrals. In contrast, Sppl exploits conditional independences, when they exist, to improve scalability (Sec. 5.1) and delivers complete, usable answers to users. Moreover, Sppl’s multi-stage workflow (Fig. 7) allows expressive computations such as translation and conditioning to be amortized over multiple datasets or queries, whereas PSI recomputes the symbolic solution from scratch each time. Hakaru [43] is a symbolic solver that delivers exact inference in a multi-stage workflow based on program transformations, and can disintegrate against a variety of base measures [44]. This paper compares
against PSI because the reference Hakaru implementation crashes or delivers incorrect or partial results on several benchmark problems [23, Table 1], and, as mentioned by the developers, does not support constructs such as arrays needed to support dozens or hundreds of observations.

**Symbolic Execution and Volume Computation:** Previous work has addressed the problem of computing the probability of a predicate by integrating a distribution defined by a program [2, 25, 55, 61]. For example, Geldenhuys et al. [25] present a probabilistic symbolic execution technique that uses model counting to compute path probabilities, assuming that all program variables are discrete and uniformly distributed. While Sppl can model a variety of distributions, due to restriction (R3) it only supports predicates that specify rectangular regions, whereas several of the aforementioned systems can (approximately) handle non-rectangular regions. More specifically, predicates in Sppl may include combinations of nonlinear transforms, each of a single variable, which are solved into linear expressions that specify unions of disjoint hyperrectangles (Appx. C.2). Table 2 shows that Sppl delivers substantial speedup on the hyperrectangular regions specified by the important class of decision trees, which are widely used in interpretable machine learning applications.

**Sum-Product Networks:** The SPFlow library [41] is an object-oriented “graphical model toolkit” in Python for constructing and querying sum-product networks. Sppl leverages a new and more general sum-product representation (Lst. 1) and solves probability and conditioning queries that are not supported by SPFlow (Thm. 4.1), which include mixed random variables, numeric transforms, and logical predicates with set-valued constraints. In addition, we introduce a novel translation strategy (Sec. 5) that allows users to specify models as generative code in a PPL (using e.g., variables, arrays, arithmetic and logical expressions, loops, branches) without having to manually manipulate low-level data structures. “Factored sum-product networks” [58] have been used as intermediate representations for converting a probabilistic program and any functional interpreter into a system of equations whose solution is the marginal probability of the program’s return value. These algorithms handle recursive procedures and leverage dynamic programming, but only apply to discrete variables, cannot handle transforms, and require solving fixed-points. Moreover, they have not been quantitatively evaluated on PPL benchmark problems.

**Weighted Model Counting/Integration:** A common approach to probabilistic inference is using algorithmic reductions from probabilistic programs to weighted-model counting (WMC) or integration (WMI) via knowledge compilation [5, 15, 19, 22, 66]. For example, Symbo [70] leverages WMI for exact inference in hybrid domains, using sentinel decision diagrams as the representation and the PSI solver to symbolically integrate over continuous variables. Dice [30] leverages WMC for scaling exact inference in discrete probabilistic programs and uses binary decision diagram representations that automatically exploit program structure to factorize inference. The representations in Dice enable substantial computation reuse for querying and/or conditioning, such as computing “all-marginal” probabilities by reusing the same compiled representation multiple times. Sppl also leverages factorization and computation reuse, but uses a different representation based on sum-product expressions that handle additional computations such as numeric transforms and continuous and mixed-type random variables.

**Probabilistic Program Synthesis:** Existing PPL synthesis systems for tabular data [13, 53] produce programs in languages that are subsets of Sppl, which enable automatic synthesis of full Sppl programs from data. Sppl can also unify and extend custom PPL query engines used in these systems for tasks such as similarity search and dependence detection [49, 50, 52]. It may also be fruitful to use structure discovery methods for time series [1, 54] or relational data [32] to synthesize Sppl programs for these domains.

## 8 Conclusion

We have presented Sppl, a new system that automatically delivers exact answers to a range of probabilistic inference queries. A key insight in Sppl is to impose restrictions on probabilistic programs that enable them to be translated to sum-product expressions, which are highly effective representations for inference. Our evaluation highlights the efficacy of Sppl on inference tasks in the literature and underscores the importance of key design decisions, including the multi-stage inference workflow and techniques used to build compact expressions by exploiting probabilistic structure. In addition to its efficacy as a standalone language, we further anticipate that Sppl could be useful as an embedded-domain-specific language within more expressive PPLs, combining the benefits of exact and approximate inference.
A Syntax of Core Calculus

The metalanguage in this paper follows that of Turbak et al. [63, Appx. A]. For completeness, Lst. 9 shows the syntax of the core calculus whose semantics are given in Lst. 1 from the main text. Prop. A.1 below establishes that distributions specified by DistInt and DistReal (Lst. 1e) with CDF $F$ can be sampled using a variant of the integral probability transform.

Proposition A.1. Let $F$ be a CDF and $r_1$, $r_2$ real numbers such that $F(r_1) < F(r_2)$. Let $U \sim$ Uniform($F(r_1), F(r_2)$) and define the random variable $X := F^{-1}(U)$. Then for all real numbers $r$,

$$
\hat{F}(r) := \Pr[X \leq r] = \begin{cases} 0 & \text{if } r < r_1 \\ \frac{F(r) - F(r_1)}{F(r_2) - F(r_1)} & \text{if } r_1 \leq r \leq r_2 \\ 1 & \text{if } r_2 < r \end{cases}
$$

(8)

Proof. Immediate from $\Pr[X \leq r] = \Pr[U \leq F(r)]$ and the uniformity of $U$ on $[r_1, r_2]$.

B Definitions of Auxiliary Functions

Sec. 3 refers to the following operations on the Outcomes domain:

- $\text{union} : \text{Outcomes}^* \rightarrow \text{Outcomes}$
- $\text{intersection} : \text{Outcomes}^* \rightarrow \text{Outcomes}$
- $\text{complement} : \text{Outcomes} \rightarrow \text{Outcomes}$

Any implementation satisfies the following invariants:

$$
\forall_i \left( v_i \Pi v_m = \text{union } v^n \iff \forall i \neq j. \text{intersection } v_i v_j = \emptyset \right),
$$

(9)

$$
\forall_i \left( v_i \Pi v_m = \text{intersection } v^n \iff \forall i \neq j. \text{intersection } v_i v_j = \emptyset \right),
$$

(10)

$$
\forall_i \left( v_i \Pi v_m = \text{complement } v \iff \forall i \neq j. \text{intersection } v_i v_j = \emptyset \right),
$$

(11)

Lst. 10 shows an implementation of the $\text{complement}$ function, which operates separately on the Real and String components; $\text{union}$ and $\text{intersection}$ are implemented similarly. Lst. 11 shows the $\text{vars}$ function, which returns the variables in a Transform or Event expression. Lst. 14 shows the $\text{negate}$ function, which returns the logical negation of an Event.

C Transforms of Random Variables

This appendix describes the Transform domain in the core calculus (expanding Lst. 1b), which is used to express numerical transformations of real random variables.

C.1 Valuation of Transforms

Lst. 17 shows the valuation function $\mathcal{T}$ which defines each $t$ as a Real function on Real. Each real function $\mathcal{T}[t]$ is defined on an input $r'$ if and only if $\mathcal{T}([r]) \in \text{domainof } t$. Lst. 18 shows the implementation of $\text{domainof}$.

C.2 Preimage Computation

Lst. 19 shows the algorithm that implements

$$
\text{preimg} : \text{Transform} \rightarrow \text{Outcomes} \rightarrow \text{Outcomes},
$$

(12)

which, as discussed in Sec. 3 of the main text, satisfies

$$
(\downarrow \text{Real Outcome} r) \in \mathcal{V} \{\text{preimg } t v\} \iff \mathcal{T}[r] (r) \in \mathcal{V} \{v\},
$$

(13)

$$
(\downarrow \text{String Outcome}s) \in \mathcal{V} \{\text{preimg } t v\} \iff (t \in \text{Identity}) \land (s \in \mathcal{V} \{v\}).
$$

The implementation of $\text{preimg}$ uses several helper functions:

- (Lst. 20) $\text{finv}$: computes the preimage of each $t \in \text{Transform}$ at a single Real.
- (Lst. 21) $\text{polyLim}$: computes the limits of a polynomial at the infinites.
- (Lst. 22) $\text{polySolve}$: computes the set of values at which a polynomial is equal to a given value (possibly positive or negative infinity).
- (Lst. 23) $\text{polyLt}$: computes the set of values at which a polynomial is less than or equal to a given value.

In addition, we assume access to a general root finding algorithm $\text{roots} : \text{Real}^+ \rightarrow \text{Real}^*$ (not shown), that returns a (possibly empty) list of roots of the degree-$m$ polynomial with specified coefficients. In the reference implementation of Sppl, the $\text{roots}$ function uses symbolic analysis for polynomials whose degree is less than or equal to two and semi-symbolic analysis for higher-order polynomials.

C.3 Example of Exact Inference on a Many-to-One Random Variable Transformation

This appendix shows how Sppl enables exact inference on many-to-one transformations of real random variables described in the previous section, where the transformation is itself determined by a stochastic branch (Fig. 4 in main text). Fig. 4a shows an Sppl program that defines a pair of random variables $X, Z$, where $X$ is normally distributed; and $Z = -X^2 + X^2 + 6X$ if $X < 1$, otherwise $Z = 5\sqrt{X} + 1$. The first plot of Fig. 4e shows the prior distribution of $X$; the middle plot shows the transformation $t$ that defines $Z = t(X)$, which is a piecewise sum of $t_d$ and $t_{else}$; and the final plot shows the distribution of $Z = t(X)$. Fig. 4b shows the sum-product expression representing this program, where the root node is a sum whose left and right children have weights 0.691... and 0.309..., which corresponds to the prior probabilities of $\{X < 1\}$ and $\{1 \leq X\}$. Nodes labeled $X \sim \mathcal{N}(\mu, \sigma)$ with an incoming directed edge from a node labeled $(r_1, r_2)$ denotes that the random variable is constrained to the interval $(r_1, r_2)$ (and similarly for closed intervals). Deterministic transformations are denoted by using red directed edges from a leaf node (i.e., $X$) to a numeric expression (e.g., $5\sqrt{X} + 11$), with the name of the transformed variable along the edge (i.e., $Z$).

Fig. 4c shows an Sppl query that conditions the program on an event $\{Z^2 \leq 4\} \cap \{Z \geq 0\}$ involving the transformed variable $Z$. The inference engine performs the following
analysis on the query:

\[
\begin{align*}
\{ Z^2 \leq 4 \} \cap \{ Z \geq 0 \} \\
\equiv \{ Z \in [0, 2] \} \quad (16) \\
\equiv \{ X \in t^{-1}([0, 2]) \} \quad \text{recall } (Z := t(X)) \quad (17) \\
\equiv \{ X \in t^{-1}([0, 2]) \} \cup \{ X \in t_{\text{else}}^{-1}([0, 2]) \} \quad (18) \\
\equiv \{-2.174... \leq X \leq -2\} \cup \{ 0 \leq X \leq .321... \} \quad (19) \\
\equiv \{ -2.174... \leq X \leq -2 \} \cup \{ 0 \leq X \leq .321... \} \quad (20)
\end{align*}
\]

Eq. (17) shows the first stage of inference, which solves any transformations in the conditioning event and yields \{0 \leq Z \leq 2\}. The conditional distribution of \(Z\) is shown in the final plot of Fig. 4f. The next step is to dispatch the simplified event to the left and right subtrees. Each subtree will compute the constraint on \(X\) implied by the event under the transformation in that branch, as shown in Eq. (19). The middle plot of Fig. (4f) shows the preimage computation under \(t_2\) from the left subtree, which gives two intervals, and \(t_{\text{else}}\) from the right subtree, which gives one interval.

The final step is to transform the prior expression (Fig. 4b) by conditioning each subtree on the intervals in Eq. (20), which gives the posterior expression (Fig. 4d). The left subtree in Fig. 4b, which originally corresponded to \{ \(X < 1\) \}, is split in Fig. 4d into two subtrees that represent the events \{-2.174... \leq X \leq -2\} and \{ 0 \leq X \leq .321... \}, respectively, and whose weights 0.159... and 0.494... are the (renormalized) probabilities of these regions under the prior distribution (first plot of Fig. 4e). The right subtree in Fig. 4b, which originally corresponded to \{ 1 \leq X \}, is now restricted to \{81/25 \leq X \leq 121/25\} in Fig. 4d and its weight 0.347... is again the (renormalized) prior probability of the region. The graph in Fig. 4d represents the distribution of \((X, Z)\) conditioned on the query in Eq. (17). The new sum-product expression be used to run further queries, such as using simulate to generate \(n\) i.i.d. random samples \(\{(x_i, z_i)\}_{i=1}^n\) from the posterior distributions in Fig. 4f or condition to condition the program on further events.

D Conditioning Sum-Product Expressions

This section presents algorithms for exact inference, that is, conditioning the distribution defined by an element of SPE (Lst. 1). Sec. D.2 focuses on a positive probability Event (Lst. 1c) and Sec. D.3 focuses on a Conjunction of equality constraints on non-transformed variables, such as \{ \(X = 3\) \} \& \{ \(Y = 4\) \} (see also Remark 4.2 in the main text). We will first prove Thm. 4.1 from the main text, which establishes that SPE is closed under conditioning on any positive probability Event. For completeness, we restate the Thm. 4.1 below.

**Theorem 4.1** (Closure under conditioning). Let \(S \in \text{SPE} \) and \(e \in \text{Event} \) be given, where \(P[S] \geq 0\). There exists an algorithm which, given \(S \) and \(e\), returns \(S' \in \text{SPE} \) such that, for all \(e' \in \text{Event} \), the probability of \(e'\) according to \(S'\) is equal to the conditional probability of \(e'\) given \(e\) according to \(S\), i.e.,

\[
P[S'](e') = P[S](e | e) \quad (5)
\]

Thm. 4.1 is a structural conjugacy property \([20]\) for the family of probability distributions defined by the SPE domain, where both the prior and posterior are identified by elements of SPE. In Sec. D.2, we present the domain function condition (Eq. (6), main text) which proves Thm. 4.1 by construction. We first discuss several preprocessing algorithms that are key subroutines used by condition.

D.1 Algorithms for Event Preprocessing

**Normalizing an Event** The \(dnf\) function (Lst. 15) converts an Event \(e\) to \(DNF\), which we define below.

**Definition D.1.** An Event \(e\) is said to be in disjunctive normal form (DNF) if and only if one of the following holds:

(D.1.1) \(e \in \text{Containment}\)

(D.1.2) \(e = e_1 \cap \cdots \cap e_m \in \text{Conjunction}\)

\[ \implies \bigwedge_{1 \leq i \leq m} e_i \in \text{Containment} \]

(D.1.3) \(e = e_1 \cup \cdots \cup e_m \in \text{Disjunction}\)

\[ \implies \bigvee_{1 \leq i \leq m} e_i \in \text{Containment} \cup \text{Conjunction} \]

Terms \(e\) and \(e_i\) in (D.1.1) and (D.1.2) are called “literals” and terms \(e_i\) in (D.1.3) are called “clauses”.

We next define the notion of an Event in “solved” DNF.

**Definition D.2.** An Event \(e\) is in solved DNF if all the following conditions hold: (i) \(e\) is in DNF; (ii) all literals within a clause \(e_i\) of \(e\) have different variables; and (iii) each literal \((t \in v)\) of \(e\) satisfies \(t \in \text{Identity} \) and \(v \notin \text{Union} \).

**Example D.3.** Using informal notation, the solved DNF form of the event \(\{X^2 \geq 9\} \cap \{|Y| < 1\}\) is a disjunction with two conjunctive clauses: \(\{(X \in (\infty, -3]\} \cup \{X \in (-\infty, 3]\}\) \cup \{\{X \in (3, \infty)\} \cap \{Y \in (-1, 1)\}\}.

Lst. 5a shows the normalize operation, which converts an Event \(e\) to solved DNF. In particular, predicates with (possibly nonlinear) arithmetic expressions are converted to predicates that contain only linear expressions (which is a property of Transform and preimg; Appx. C); e.g., as in Eqs. (17)–(20).

**Proposition D.4.** \(\forall e \in \text{Event}, \mathbb{E}[e] = \mathbb{E}(dnf e)\) and denotations of Union (Lst. 1a) and Disjunction (Lst. 1c).

**Disjoining an Event** Suppose that \(e \in \text{Event}\) is in DNF and has \(m \geq 2\) clauses. A key inference subroutine is to rewrite \(e\) in solved DNF (Def. D.2) where all the clauses are disjoint.
Definition D.5. Let \( e \) \in Event be in DNF. Two clauses \( e_i \) and \( e_j \) of \( e \) are said to be disjoint if both \( e_i \) and \( e_j \) are in solved DNF and at least one of the following conditions holds:

\[
\exists x \in (\text{vars } e_i), \quad \mathbb{E} [e_{ix}] x \equiv \emptyset \quad (21)
\]

\[
\exists x \in (\text{vars } e_j), \quad \mathbb{E} [e_{jx}] x \equiv \emptyset \quad (22)
\]

\[
\exists x \in (\text{vars } e_i) \cap (\text{vars } e_j), \quad \mathbb{E} [e_{ix} \land e_{jx}] x \equiv \emptyset \quad (23)
\]

where \( e_{ix} \) denotes the unique literal of \( e_i \) that contains variable \( x \) (for each \( x \in \text{vars } e_i \)), and similarly for \( e_j \).

Lst. 16 shows the disjoint? procedure, which given a pair of clauses \( e_i \) and \( e_j \) that are in solved DNF (as produced by normalize), returns true if and only if one of the conditions in Def. D.5 hold. Lst. 5b presents the main algorithm disjoint, which decomposes an arbitrary Event \( e \) into solved DNF whose clauses are mutually disjoint. Prop. D.6 establishes the correctness and worst-case complexity of disjoint.

Proposition D.6. Let \( e \) be an Event with \( h := |\text{vars } e| \) variables, and suppose that \( e_1 \cup \cdots \cup e_m := \text{(normalize } e) \) has exactly \( m \geq 1 \) clauses. Put \( \tilde{e} := \text{(disjoint } e) \). Then:

\[
\begin{align*}
&D.6.1 \quad \tilde{e} \text{ is in solved DNF.} \\
&D.6.2 \quad \forall 1 \leq i \neq j \leq \ell. \quad \text{disjoint? (} e_i, e_j \text{).} \\
&D.6.3 \quad \mathbb{E} [\tilde{e}] = \mathbb{E} [e]. \\
&D.6.4 \quad \text{The number } \ell \text{ of clauses in } \tilde{e} \text{ satisfies } \ell \leq (2m - 1)^h.
\end{align*}
\]

Proof. Suppose first that (normalize } e) has \( m = 1 \) clause \( e_1 \). Then \( \tilde{e} = e_1 \), so (D.6.1) holds since \( e_1 = \text{normalize } e \); (D.6.2) holds trivially; (D.6.3) holds by Prop. D.4; and (D.6.4) holds since \( \ell = (2 - 1)^1 = 1 \). Suppose now that (normalize } e) has \( m > 1 \) clauses. To employ set-theoretic reasoning, fix some \( x \in \text{Var} \) and define \( \mathbb{E}' [e] := \forall x \mathbb{E} [e] x \subseteq \text{Outcome} \), for all \( e \in \text{Event} \). We have

\[
\begin{align*}
\mathbb{E}' [e_1 \cup \cdots \cup e_m] &= \bigcup_{i=1}^m \mathbb{E}' [e_i] \quad (25) \\
&= \bigcup_{i=1}^m \left( \mathbb{E}' [e_i] \land \neg \bigcup_{j \neq i} \left( \mathbb{E}' [e_j] \right) \right) \quad (26) \\
&= \bigcup_{i=1}^m \left( \mathbb{E}' [e_i] \land \neg \bigcup_{j \neq i} \left( \mathbb{E}' [e_j] \right) \right) \quad (27) \\
&= \bigcup_{i=1}^m \left( \mathbb{E}' [e_i] \land \neg \bigcup_{j \neq i} \left( \mathbb{E}' [e_j] \right) \right) \quad (28) \\
&= \bigcup_{i=1}^m \left( \mathbb{E}' [e_i] \land \neg \bigcup_{j \neq i} \left( \mathbb{E}' [e_j] \right) \right) \quad (29)
\end{align*}
\]

where we define for each \( i = 1, \ldots, m \),

\[
k(i) := \{ 1 \leq j \leq i - 1 \mid \mathbb{E}' [e_i] \land \mathbb{E}' [e_j] \neq \emptyset \}.
\]

Eq. (29) follows from the fact that for any \( i = 1, \ldots, m \) and \( j < i \), we have

\[
j \not\in k(i) \implies \left[ \left( \mathbb{E}' [e_i] \land \neg \mathbb{E}' [e_j] \right) \equiv \mathbb{E}' [e_i] \right].
\]

As negate (Lst. 14) computes set-theoretic complement \( \neg \) in the Event domain and \( j \not\in k(i) \) if and only if (disjoint? \( e_j, e_i \)), it follows that the Events \( e'_i := e_i \land \tilde{e}_i \) (\( i = 2, \ldots, m \)) in Eq. (24c) are pairwise disjoint and are also disjoint from \( e_1 \), so that \( \mathbb{E}' [e] = \mathbb{E} [e_1 \cup e'_2 \cup \cdots \cup e'_m] \). Thus, if disjoint halts, then all of (D.6.1)–(D.6.3) follow by induction.

We next establish that disjoint halts by upper bounding the number of clauses \( \ell \) returned by any call to disjoint. Recalling that \( h := |\text{vars } e| \), we assume without loss of generality that all clauses \( e_i (i = 1, \ldots, n) \) in Eq. (24a) have the same variables \( x_1, \ldots, x_k \), by “padding” each \( e_i \) with vacuously true literals of the form \( (\text{Id}(x_j) i \in \text{Outcomes}) \). Next, recall that clause \( e_i \) in Eq. (24a) is in solved DNF and has \( m_i \geq 1 \) literals \( e_{ij} = (\text{Id}(x_{ij}) i \in \text{Union}) \) where \( v_{ij} \notin \text{Union} \) (Def. D.2). Thus, \( e_i \) specifies exactly one hyperrectangle in \( h \)-dimensional space, where \( v_{ij} \) is the “interval” (possibly infinite) along the dimension specified by \( x_{ij} \) in literal \( e_{ij} \) \((i = 1, \ldots, m; j = 1, \ldots, m_i) \). A sufficient condition to produce the worst-case number of pairwise disjoint primitive sub-hyperrectangles that partition the region \( e_1 \cup \cdots \cup e_m \) is when the previous clauses \( e_1, \ldots, e_{m-1} \) (i) are pairwise disjoint (Def. D.5); and (ii) are strictly contained in \( e_m \), i.e., \( \forall x, \mathbb{E} [e_j] \subseteq \mathbb{E} [e_m], (j = 1, \ldots, m - 1) \). If these two conditions hold, then disjoint partitions the interior of the \( h \)-dimensional hyperrectangle specified by \( e_m \) into no more than \( 2(m - 1)^h \) sub-hyperrectangles that do not intersect one another (and thus, produce no further recursive calls), thereby establishing (D.6.4). \( \square \)

Example D.7. The left panel in Fig. 9 shows \( m = 4 \) rectangles in Real \( \times \) Real. The right panel shows a grid (in red) with \( (2m - 1)^2 = 49 \) primitive rectangular regions that are pairwise disjoint from one another and whose union over-approximates the union of the 4 rectangles. In this case, 29 of these primitive rectangular regions are sufficient (but excessive) to exactly partition the union of the rectangles into a disjoint union. No more than 49 primitive rectangles are ever needed to partition any 4 rectangles in Real, and this bound is tight. The bound in (D.6.4) generalizes this idea to hyperrectangles that live in \( h \)-dimensional space.

![Conditioning Region](conditioning_region.png)

![Partition into Rectangles](partition_into_rectangles.png)

Figure 9. Example illustrating the upper bound (D.6.4) on the number of disjoint rectangles in a worst-case partition of a conditioning region in the two-dimensional Real plane.

Remark D.8. When defining \( \tilde{e} \) in Eq (24b) of disjoint, ignoring previous clauses that are disjoint from \( e_i \) is essential for disjoint to halt, so as to avoid recursing on a primitive sub-rectangle in the interior. That is, filtering out such clauses ensures that disjoint makes a finite number of recursive calls.
D.2 Algorithms for Conditioning Sum-Product Expressions on Positive Measure Events

Having established the key background details, we now prove Thm. 4.1 from the main text, which establishes the closure under conditioning property of the SP domain.

Proof of Theorem 4.1. We establish Eq. (5) by defining

\[
\text{condition} : \text{SPE} \rightarrow \text{Event} \rightarrow \text{SPE}
\]

which satisfies

\[
\mathbb{P} \left[ \left( \text{condition } S \right) e' \right] = \frac{\mathbb{P} \left[ S \right] \left( e \cap e' \right)}{\mathbb{P} \left[ S \right] e}
\]

for all \( e' \in \text{Event} \) and \( e \in \text{Event} \) for which \( \mathbb{P} \left[ S \right] e > 0 \).

We will define \text{condition} separately for each of the three constructors Leaf, Sum, and Product from Lst. 9f. The proof is by structural induction, where Leaf is the base case and Sum and Product are the recursive cases.

Conditioning Leaf

Lst. 6a shows the base cases of \text{condition}.

The case of \text{d} \in \text{DistInt} is straightforward. For \text{d} \in \text{DistReal}, if the intersection (defined in second line of Lst. 6a) of \text{v} with the support of \text{d} is an interval \((b'_1, r'_1), (r'_2, b'_2))\), then it suffices to return a Leaf restricting \text{d} to the interval. If the intersection is a Union \(v_1, \ldots, v_m\) (recall fro Eq. (13) that \text{intersection} ensures the \(v_i\) are disjoint), then the conditioned \text{SP} is a Sum, whose \(i\)th child is obtained by recursively calling \text{condition} on \(v_i\) and \text{ith} (relative) weight is the probability of \(v\) under \(d\), since, for any new \(v'\) \in \text{Outcomes}, we have

\[
\mathbb{D} \left[ d' \right] \left[ \left( \text{intersection } v' \right) (v_1, \ldots, v_m) \right] = \frac{\mathbb{D} \left[ d \right] (v_1, \ldots, v_m)}{\mathbb{D} \left[ d \right] (v_1, \ldots, v_m)}
\]

Eq. (33) follows from the additivity of \(\mathbb{D} \left[ d \right] \). The plots of \(X\) in Figs. 4e and 4f illustrate the identity in Eq. (33), where conditioning the unimodal normal distribution results in a mixture of three restricted normals whose weights are given by the relative prior probabilities of the three regions.

For \(d \in \text{DistReal},\) if the positive probability Outcomes are \(\{r_1, \ldots, r_m\}\), then the conditioned \text{SP} is a Sum of “delta”-CDFs whose atoms are located on the integers \(r_i\) and weights are the (relative) probabilities \(\mathbb{D} \left[ d \right] (\{r_i\}) = (i = 1, \ldots, m)\). Since the atoms of \(F\) for DistReal are integers, it suffices to restrict \(F\) to the interval \((r_i - 1/2, r_i)\), for each \(r_i\) with a positive weight. Correctness again follows from Eq. (33), since finite sets are unions of disjoint singleton sets. For other positive probability Outcomes, the conditioning procedure \text{DistInt} is the same as that for \text{DistReal}.
We next establish Thm. 4.3 from the main text, which gives
conditioning holds for computing Event probabilities \( \mathbb{P}[S | e] \) in Lst. 1f and probability densities \( \mathbb{P}_{0}[S] e \), Lst. 1d).

\[ \mathbb{P}[(S_1 \cdot \cdots \cdot S_m) (e \cap e') = P_{m=1}^{\sum} w_i P(S_i) (e \cap e') \]

(34)

(35)

(36)

(37)

where Eq. (36) has applied Eq. (32) inductively for each \( S_i \). Eqs. (35)–(36) assume for simplicity that \( P(S_i) > 0 \) for each \( i = 1, \ldots, m \), whereas Lst. 6a does not make this assumption.

**Conditioning Sum** Lst. 6b shows condition for \( S \in \mathrm{Sum} \). Recalling the denotation \( P[S] \) for \( S \in \mathrm{Sum} \) in Lst. 1f, the correctness follows from the following properties:

**Conditioning Product** Lst. 6c how condition operates on \( S \in \mathrm{Product} \). The first step is to invoke \( \text{disjoin} \) to rewrite \( \text{dnf} e \) as \( \ell \geq 1 \) disjunct clauses \( e_1 \cup \cdots \cup e_{\ell} \) (recall from Prop. D.6 that \( \text{disjoin} \) is semantics-preserving). The first pattern in the \textit{match} statement corresponds \( \ell = 1 \), and the result is a new Product, where the \( i \)th child is conditioned on the literals of \( e_1 \) whose variables are contained in \( \text{scope} S_1 \) (if any). The second pattern returns a Sum of Product, since

\[ \sum_{i=1}^{\ell} P(S_1 \cdot \cdots \cdot S_m) (e_1 \cap e') \]

(38)

(39)

(40)

(41)

(42)

(43)

Eq (42) follows from the induction hypothesis Eq. (32) and \( (\text{disjoin} e_1) \equiv e_1 \) (idempotence), so that \( (\text{disjoin} e_1 \cap e') \equiv (\text{disjoin} e_1) \cap (\text{disjoin} e') \equiv e_1 \cap (\text{disjoin} e') \). Thm. 4.1 is thus established.

Fig. 5 in the main text shows an example of the closure property from Thm. 4.1, where conditioning on a hyperrectangle changes the structure of the SPE from a Product into a Sum-of-Product. The algorithms in this section are the first to describe probabilistic inference and closure properties for conditioning an SPE on a query that involves transforms of random variables and predicates with set-valued constraints. We next establish Thm. 4.3 from the main text, which gives a sufficient condition for the runtime of \textit{condition} (Lst. 6) to scale linearly in the number of nodes in \( S \); identical results hold for computing Event probabilities \( \mathbb{P}[S | e] \), Lst. 1f) and probability densities \( \mathbb{P}_{0}[S] e \), Lst. 1d).

**Theorem 4.3.** The runtime of \( (\text{condition} S e) \) scales linearly in the number of nodes in the graph representing \( S \) whenever \( e \) is a single Conjunction \( (t_1 \in v_1) \cap \cdots \cap (t_m \in v_m) \) of Containment constraints on non-transformed variables.

**Proof.** First, if \( S \) is a Sum (Lst. 6b) with \( m \) children then \( (\text{condition} e) \) makes no more than \( m \) subcalls to \text{condition} (one for each child), and if \( S \) is a Leaf (Lst. 6a) then there are zero subcalls, independently of \( e \). Since each node has exactly one parent, we can conclude that each node in \( S \) is visited exactly once by showing that the hypothesis on \( e \) implies that for any \( S \in \mathrm{Product} \) (Lst. 6c) with \( m \) children, there are at most \( m \) subcalls to \text{condition} from which we (each node has exactly one parent). Suppose that \( (\text{disjoin} e) \) returns a single Conjunction. Then the first pattern of the \textit{match} statement in Lst. 6c is matched (one \( h \)-dimensional rectangle), resulting in \( m \) subcalls to \text{condition}. Thus, each node in \( S \) is visited (at most) once by \text{condition}. To complete the proof, note that the hypothesis that \( e \) specifies a single Conjunction \( (t_1 \in v_1) \cap \cdots \cap (t_m \in v_m) \) of Containment constraints on non-transformed variables is sufficient for \( (\text{disjoin} e) \) to return a single Conjunction. \( \square \)

**D.3 Conditioning Sum-Product Expressions on Measure Zero Equality Constraints**

Recall from Remark 4.2 in the main text that SPE is also closed under conditioning on a Conjunction of possibly measure zero equality constraints of non-transformed variable, such as \( \{X = 3, Y = \pi, Z = \text{foo}\} \). In this section, we describe the conditioning algorithm for this case, which is implemented by

\[ \text{condition}_{0} : \text{SPE} \rightarrow \text{Event} \rightarrow \text{SPE}, \]

(44)

where \( e \in \text{Event} \) satisfies the following requirements with respect to \( S \in \text{SPE} \):

1. Either \( e \equiv (\text{Id}(x) \in \{rs\}) \) or \( e \) is a Conjunction of such literals, where \( \equiv \) here denote syntactic (not semantic) equivalence.

2. Every \( \text{Var} x \) in each literal of \( e \) is a non-transformed variable; i.e., for each Leaf expression \( S \) such that \( x \in \text{scope} S \), we have \( S \equiv \text{Leaf}(x d \sigma) \), for some \( d \) and \( \sigma \).

With these requirements on \( e \), Lst. 7 presents the implementation of \( \text{condition}_{0} \), leveraging the generalized density semantics from Lst. 1d in the main text. The inference rules closely match those for standard sum-product networks, except for the fact that a density from \( \mathbb{P}_{0}[S] \) is a pair, whose first entry is the number of continuous distributions participating in the weight of the Event \( e \) which must be correctly accounted for by \( \text{condition}_{0} \). In the reference implementation of \( \text{Sppl} \), \textit{condition} invokes \textit{condition}_{0} analogously to the \textit{prob} query, which returns probabilities using the distribution semantics \( \mathbb{P} \) in Lst. 9e. \textit{Sppl} also includes the \textit{density} query, which returns densities using the generalized semantics \( \mathbb{P}_{0} \) in Lst. 1d.
condition_0 Leaf(x d σ) (Id(x) in (rs)) := match d
    ▷ Dist(F r1 r2) ⇒ match rs
    ▷ r ⇒ match (F [Leaf(x d σ)] (Id(x) in (rs)))
    ▷ (1, 0) ⇒ undefined
    ▷ else let F be (lr’ . 1 | 0 ≤ r’) in Dist(I (F (r - 1/2) r))
    ▷ s ⇒ undefined
    ▷ else ⇒ condition Leaf(x d σ) (Id(x) in (rs))

(a) Conditioning Leaf

condition_0 ((S_1 w_1) ⊗ · · · ⊗ (S_m w_m)) \( \{ t_{i=1}^n (Id(x_i) in (rs_i)) \} \) :=
let \( i \leq m \) \( d_i, p_i \) be \( P_0 [S_i] \) \( \{ t_{i=1}^n (Id(x_i) in (rs_i)) \} \)
in if \( V_{i \leq m}, p_i = 0 \) then undefined
else let \( i \leq m \) \( w_i' \) be \( w_i p_i \)
in let \( d' \) be \( \min \{ d_i \mid 1 \leq i \leq m, 0 < p_i \} \)
in let \( \{ n_1, \ldots, n_k \} \) be \( \{ n \mid 0 < w_i', d_i = d' \} \)
in let \( i \leq l \leq m \) \( S'_i \) be \( \{ \text{condition}_0 S_{n_i} \} \{ t_{i=1}^n (Id(x_i) in (rs_i)) \} \)
in if \( k = 1 \) then \( S' \) else \( \text{else}_i (S'_i w_i) \)

(b) Conditioning Sum

condition_0 (S_1 ⊗ · · · ⊗ S_m) \( \{ t_{i=1}^n (Id(x_i) in (rs_i)) \} \) :=
let \( i \leq m \) \( x, t_1, \ldots, t_m \) \( \{ \text{scope} S_i \} \)
▷ \( \{ n_1, \ldots, n_k \} \) ⇒ condition_0 S_{n_i} \( \{ t_{i=1}^n (Id(x_i) in (rs_i)) \} \)
in \( S'_1 \otimes \cdots \otimes S'_m \)

(c) Conditioning Product

Listing 7. Implementation of condition_0 for Leaf, Sum, and Product expressions using density semantics in Lst. 1d.

E Translating Sum-Product Expressions to Sppl Programs

Lst. 3 in Sec. 5 presents the relation \( \rightarrow_{\text{Sppl}} \), that translates \( C \in \text{Command} \) (i.e., Sppl source syntax, Lst. 2) to a sum-product expression \( S \in \text{SPE} \) in the core language (Lst. 9). Lst. 8 defines a relation \( \rightarrow_{\text{Sppl}} \) that reverses the \( \rightarrow_{\text{SPE}} \) relation: it converts expression \( S \in \text{SPE} \) to \( C \in \text{Command} \). Briefly, (i) a Product is converted to a sequence Command; (ii) a Sum is converted to an if-else Command; and (iii) a Leaf is converted to a sequence of sample (\( \sim \)) and transform (\( \sim \)). The symbol \( \|$ \) (whose definition is omitted in the (LEAF) rule converts semantic elements such as \( d \) in Distribution and \( t \) in Transform from the core calculus (Lst. 1) to an Sppl expression \( E \in \text{Expr (Lst. 2)} \) in a straightforward way, e.g.,

\[
(\text{Poly}(\text{Id}(x \ 1 \ 2 \ 3)) \|$ (1 + 2*x + 3*x**2)). \quad (45)
\]

It is easy to see that chaining \( \rightarrow_{\text{Sppl}} \) (Lst. 3) and \( \rightarrow_{\text{SPE}} \) (Lst. 8) for a given Sppl program does not preserve either Sppl or core syntax, that is

\[
\begin{align*}
(C \rightarrow_{\text{SPE}} S) &\rightarrow_{\text{Sppl}}^* C' \quad \text{does not imply } C = C' \\
(C \rightarrow_{\text{SPE}} S) &\rightarrow_{\text{Sppl}}^* C' \rightarrow_{\text{SPE}}^* S' \quad \text{does not imply } S = S'.
\end{align*}
\]

The symbol \( C \rightarrow_{\text{SPE}} S \) means \( (C, S_0) \) translates to \( S \in \text{zero or more steps of } \rightarrow_{\text{SPE}}, \) where \( S_0 \) is an “empty” SP used for the initial translation step, and similarly for \( \rightarrow_{\text{Sppl}}^* .\)

\[
\begin{align*}
&d \Op D(E), t_1 \Op E_1, \ldots, t_m \Op E_m \\
&\rightarrow_{\text{Sppl}} x \sim D(E); x \Op E_1; \ldots; x_m \Op E_m \\
&\rightarrow_{\text{SPE}} S \Op \text{SPE} C_1; \ldots; C_m
\end{align*}
\]

Listing 8. Translating an element of SPE (Lst. 9f) to an Sppl command \( C \) (Lst. 2).

Instead, it can be shown that \( \rightarrow_{\text{Sppl}} \) is a semantics-preserving inverse of \( \rightarrow_{\text{SPE}} \), in the sense that for all \( e \in \text{Event} \)

\[
((C \rightarrow_{\text{SPE}} S) \rightarrow_{\text{Sppl}}^* C') \rightarrow_{\text{SPE}} S' \implies \mathbb{P} [S] \in \mathbb{P} [S'] \ e. (46)
\]

Eq. (46) establishes a formal semantic correspondence between the Sppl language and the class of sum-product expressions: each Sppl program admits a representation as an SPE, and each valid element of SPE that satisfies conditions (C1)–(C5) expression corresponds to some Sppl program.

Thus, in addition to synthesizing full Sppl programs from data using the PPL synthesis systems [13, 53] mentioned in Sec. 7, it is also possible with the translation strategy in Lst. 8 to synthesize Sppl programs using the wide range of techniques for learning the structure and parameters of sum-product networks [26, 37, 62, 65]. With this approach, Sppl (i) provides users with a uniform representation of existing sum-product networks as generative source code in a formal PPL (Lst. 2); (ii) allows users to extend these baseline programs with modeling extensions supported by the core calculus (Lst. 1), such as predicates for decision trees and numeric transformations; and (iii) delivers exact answers to an extended set of probabilistic inference queries (Sec. 4) within the modular and reusable workflow from Fig. 1.
\[ r \in \text{Real} \cup [-\infty, \infty) \]

\[ s \in \text{String} \equiv \text{Char}^* \]

(a) Basic Sets

\[ rs \in \text{Outcome} \Rightarrow \text{Real + String} \]

\[ v \in \text{Outcomes} \Rightarrow \emptyset \quad [\text{Empty}] \]

\[ (r_1 \ldots r_m) \] [\text{FiniteStr}] \]

\[ ((h_1, r_1) (r_2 b_2)) \] [\text{Interval}] \]

\[ v_1 \sqcup \cdots \sqcup v_m \] [\text{Union}] \]

(b) Outcomes

\[ t \in \text{Transform} \]

\[ \equiv \text{Id}(x) \quad [\text{Identity}] \]

\[ \text{Reciprocal}(t) \quad [\text{Reciprocal}] \]

\[ \text{|Abs}(t) \quad [\text{|AbsValue}] \]

\[ \text{Root}(t n) \quad [\text{Radical}] \]

\[ \text{Exp}(r) \quad [\text{Exponent}] \]

\[ \text{Log}(r) \quad [\text{Logarithm}] \]

\[ \text{Poly}(t r_0 \ldots r_m) \quad [\text{Polynomial}] \]

\[ \text{Piecewise}(t e_1) \quad [\text{Piecewise}] \]

(c) Transformations

\[ F \in \text{CDF} \subseteq \text{Real} \rightarrow [0, 1] \]

\[ \Rightarrow \text{Norm}(r_1, r_2) | \text{Poisson}(r) | \text{Binom}(n, w) \ldots \]

where \( F \) is càdlàg:

\[ \lim_{r \rightarrow -\infty} F(r) = 1, \quad \lim_{r \rightarrow +\infty} F(r) = 0; \]

\[ \text{and } F^{-1}(u) := \inf \{ r \mid u \leq F(r) \}. \]

\[ d \in \text{Distribution} \]

\[ \Rightarrow \text{Dist}\{F r_1 r_2\} \quad [\text{DistReal}] \]

\[ \text{Dist}\{F (r_1 r_2)\} \quad [\text{DistInt}] \]

\[ \text{Dist}\{(s_1 w_1) \ldots (s_m w_m)\} \quad [\text{DistStr}] \]

(e) Primitive Distributions

\[ \sigma \in \text{Environment} \Rightarrow \text{Var} \rightarrow \text{Transform} \]

\[ S \in \text{SPE} \]

\[ \Rightarrow \text{Leaf}(x d \sigma) \quad [\text{Leaf}] \]

\[ |(S_i w_i) \sqcup \cdots \sqcup (S_m w_m)| \quad [\text{Sum}] \]

\[ |S_i \sqcup \cdots \sqcup S_m| \quad [\text{Product}] \]

(f) Sum-Product

\[ \text{complement} \{ s_1 \ldots s_m \} := \{ s_1 \ldots s_m \}^{-b} \]

\[ \text{complement} \{(b_1 r_1) (r_2 b_2)\} := (\#f - \infty) (r_1 \#bf) \cap (\#bf r_2) (\#f \#bf) \]

\[ \text{complement} \{r_1 \ldots r_m\} := ((\#f - \infty) (r_1 \#t)) \]

\[ \cap (\#bf r_m) (\#f \#bf) \]

\[ \text{complement} \emptyset := \{(\#f - \infty) (\#f \#bf)\} \]

Listing 9. Core calculus.

Listing 10. Implementation of \textit{complement} on the sum domain Outcomes.

vars : (Transform + Event) \rightarrow \mathcal{P}(\text{Vars})

vars te = match te

\[ \Rightarrow t \Rightarrow \text{match } t \]

\[ \Rightarrow \text{Id}(x) \Rightarrow \{x\} \]

\[ \Rightarrow \text{Root}(t n) \mid \text{Exp}(t r) \mid \text{Log}(t r) \mid \text{Abs}(t) \]

\[ \mid \text{Reciprocal}(t) \mid \text{Poly}(t r_0 \ldots r_m) \]

\[ \Rightarrow \text{vars } t \]

\[ \Rightarrow \text{Piecewise}(t e_1) \Rightarrow \text{vars } t \]

\[ \Rightarrow (t \text{ in } v) \Rightarrow \text{vars } t \]

\[ \Rightarrow (e_1 \sqcap \cdots \sqcap e_m) \mid (e_1 \sqcup \cdots \sqcup e_m) \Rightarrow \text{vars } e_i \]

Listing 11. Implementation of \textit{vars}, which returns the variables in a Transform or Event.

scope : SPE \rightarrow \mathcal{P}(\text{Var})

\[ \text{scope } (x d \sigma) := \text{dom}(\sigma) \]

\[ \text{scope } (S_1 \sqcup \cdots \sqcup S_m) := \cup_{i=1}^m (\text{scope } S_i) \]

\[ \text{scope } ((S_1 w_1) \sqcup \cdots \sqcup (S_m w_m)) := (\text{scope } S_i) \]

Listing 12. Implementation of \textit{scope}, which returns the set of variables in an element of \textit{SPE}.
subsenv : Event → Environment → Event
subsenv e σ := let \{x, x_1, \ldots, x_m\} = dom(σ)
    in let e_1 be subs e x_m σ(x_m)
        \ldots
    in let e_m be subs e_{m-1} x_1 σ(x_1)
    in e_m

Listing 13. Implementation of subsenv, which rewrites e as an Event e' on one variable x.

negate : Event → Event
negate (t in v) := match (complement v)
    ▷ v_1 \sqcup \cdots \sqcup v_m ⇒ (t in v_1) \sqcup \cdots \sqcup (t in v_m)
    ▷ v ⇒ (t in v)


dnf : Event → Event
dnf (t in v) := (t in v)
dnf e_1 \sqcup \cdots \sqcup e_m := \sqcup_{i=1}^m(dnf e_i)
dnf e_1 \sqcap \cdots \sqcap e_m := let \{e'_j, n_{j,i}\}_{i=1}^{m} \text{ be } dnf e_i
    in \left\{ \begin{array}{l}
    \text{intersection } v_1, n_1, i = (\text{intersection } v_2, n_2, i) = \emptyset
    \end{array} \right\}

Listing 15. dnf converts and Event to DNF (Def. D.1).

disjoint? : Event × Event → Boolean
disjoint? \langle e_1, e_2 \rangle := match \langle e_1, e_2 \rangle
    ▷ \langle \sqcap_{i=1}^{m_1}(\text{Id}(x_{1,i}) \in v_{1,i}), \sqcap_{i=1}^{m_2}(\text{Id}(x_{2,i}) \in v_{2,i}) \rangle
    ⇒ [\exists_{1 \leq i \leq 2}. \exists_{1 \leq j \leq m_i}. \text{intersection } v_{1,j} = \emptyset] \lor [\text{let } \{\langle n_{1,i}, n_{2,i}\rangle\}_{i=1}^k \text{ be } \{\langle i, j \rangle \mid x_{1,i} = x_{2,j}\}\text{ in } (\exists_{1 \leq i \leq k}. \text{intersection } v_{1,n_{1,i}}, v_{2,n_{2,i}}) = \emptyset]
    ▷ else ⇒ undefined

Listing 16. disjoint? returns #t if two Events are disjoint (Def. D.5).
\[ T : \text{Transform} \rightarrow (\text{Real} \rightarrow \text{Real}) \]

\[ T[\text{Id}(x)] := \lambda r'. r' \]
\[ T[\text{Reciprocal}(t)] := \lambda r'. 1/(T[t](r')) \]
\[ T[\text{Abs}(t)] := \lambda r'. |T[t](r')| \]
\[ T[\text{Root}(t n)] := \lambda r'. \sqrt[n]{T[t](r')} \]
\[ T[\text{Exp}(t r)] := \lambda r'. r^{(T[t](r'))} \quad \text{(iff } 0 < r) \]
\[ T[\text{Log}(t r)] := \lambda r'. \log_r(T[t](r')) \quad \text{(iff } 0 < r) \]

\[ T[\text{Poly}(r_0 \ldots r_m)] := \lambda r'. \sum_{i=0}^{m} r_i (T[t](r'))^i \]
\[ T[\text{Piecewise}((t_i e_i)_{i=1}^m)] := \lambda r'. \text{if } \left( \left( \downarrow \text{Real Outcome } r' \right) \in \forall \left( \exists \left( \forall x \in \{t_i\} \right) \right) \right) \text{ then } T[t_i] r' \]
\[ \text{else if } \ldots \]
\[ \text{else if } \left( \left( \downarrow \text{Real Outcome } r' \right) \in \forall \left( \exists \left( \forall x \in \{e_m\} \right) \right) \right) \text{ then } T[t_m] r' \]
\[ \text{else undefined} \]
\[ \text{iff (vars } t_i) = \ldots = (\text{vars } t_m) \]
\[ = (\text{vars } e_1) = \ldots = (\text{vars } e_m) =: \{x\} \]

Listing 17. Semantics of Transform.

domainof : Transform \rightarrow Outcomes

\[ \text{domainof } \text{Id}(x) := ((\#f -\infty) (\infty f)) \]
\[ \text{domainof } \text{Reciprocal}(t) := ((\#f 0) (\infty f)) \]
\[ \text{domainof } \text{Abs}(t) := ((\#f -\infty) (\infty f)) \]
\[ \text{domainof } \text{Root}(t n) := ((\#f 0) (\infty f)) \]
\[ \text{domainof } \text{Exp}(t r_0) := ((\#f -\infty) (\infty f)) \]
\[ \text{domainof } \text{Log}(t r_0) := ((\#f 0) (\infty f)) \]
\[ \text{domainof } \text{Poly}(r_0 \ldots r_m) := ((\#f -\infty) (\infty f)) \]
\[ \text{domainof } \text{Piecewise}((t_i e_i)_{i=1}^m) := \text{union} \left[ \left( \text{intersection} \left( \text{domainof } t_i \right) \left( \exists \left( \forall x \in \{e\} \right) \right) \right) \right] \]

where \{x\} := \text{vars } t_1

Listing 18. domainof returns the Outcomes on which a Transform is defined.
preimg \ t \ v := \ preimage^L \ t \ (\text{intersection (domain of) } t \ v)
preimage^L \ \text{Id} \ v := v
preimage^L \ t \ \varnothing := \varnothing
preimage^L \ t \ \{v_1 \ | \ \cdots \ | \ v_m\} := \text{union} (preimg \ t \ v_1) \ \ldots \ (preimg \ t \ v_m)
preimage^L \ t \ \{r_1 \ldots r_m\} := \text{preimg} \ t \ (\text{union} (finv \ t \ r_1) \ \ldots \ (finv \ t \ r_m))
preimage^L \ t \ ((\{b_{\text{left}} \ n_{\text{left}}\} (r_{\text{right}} b_{\text{right}})) := \text{match} \ t
\triangleright \text{Radical} (t' n) | \text{Exp (t' r)} | \text{Log (t' r)} \Rightarrow \text{let} \ r'_{\text{left}} \ be \ finv \ t \ n_{\text{left}}
in \text{let} \ r'_{\text{right}} \ be \ finv \ t \ n_{\text{right}}
in \text{preimg} \ t' ((\{b_{\text{left}} r'_{\text{left}}\} (r_{\text{right}} b_{\text{right}}))
\triangleright \text{Abs (t')} \Rightarrow \text{let} \ r'_{\text{pos}} \ be ((\{b_{\text{left}} r'_{\text{left}}\} (r_{\text{right}} b_{\text{right}}))
in \text{let} \ r'_{\text{neg}} \ be ((\{b_{\text{right}} \neg r_{\text{right}}\} \neg n_{\text{left}} b_{\text{left}}))
in \text{preimg} \ t' ((\{b_{\text{left}} r'_{\text{left}}\} (r_{\text{right}} b_{\text{right}}))
\triangleright \text{Reciprocal} (t') \Rightarrow \text{let} \ [r'_{\text{left}}, r'_{\text{right}}] \ be \text{if} (0 \leq n_{\text{left}} < n_{\text{right}})
\text{then} \{\text{if} (0 < n_{\text{left}}) \text{ then }1 / n_{\text{left}} \text{ else } \infty,
\text{if} (r_{\text{right}} < \infty) \text{ then }1 / n_{\text{right}} \text{ else } 0,
\text{else} \text{if} (\infty < n_{\text{left}}) \text{ then }1 / n_{\text{left}} \text{ else } 0,
\text{if} (r_{\text{right}} < 0) \text{ then }1 / n_{\text{right}} \text{ else } -\infty\}
in \text{preimg} \ t' ((\{b_{\text{right}} r'_{\text{right}}\} (r_{\text{left}} b_{\text{left}}))
\triangleright \text{Polynomial} (t \ r_0 \ldots r_m) \Rightarrow \text{let} \ r'_{\text{left}} \ be \text{polyLte} \ b_{\text{left}} n_{\text{left}} r_0 \ldots r_m
\text{in let} \ r'_{\text{right}} \ be \text{polyLte} \ b_{\text{right}} r_{\text{right}} r_0 \ldots r_m
\text{in} \text{preimg} \ t' (\text{intersection} \ r'_{\text{right}} \ (\text{complement} \ r'_{\text{left}}))
\triangleright \text{Piecewise} ((t_i e_i)_{i=m}^0) \Rightarrow \text{let} \ t_i \ be \text{preimg} \ t_i \ ((\{b_{\text{left}} r'_{\text{left}}\} (b_{\text{right}} r_{\text{right}}))
in \text{let} t_{\sum i \leq m} \ be \text{intersection} \ r'_{\text{left}} \text{ (union) } t_{\sum i \leq m} \text{ where} \ (x) := \text{vars} t_i
\triangleright \text{Outcomes}

Listing 19. \text{preimg} \ computes \ the \ generalized \ inverse \ of \ a \ many-to-one \ Transform.

\text{finv : Transform} \rightarrow \text{Real} \rightarrow \text{Outcomes}
\text{finv} \ \text{Id} \ (x) \ r := \{r\}
\text{finv} \ \text{Reciprocal} \ (t \ r) := \{\text{if} (r = 0) \text{ then } (-\infty, \infty) \text{ else } \{1 / r\}\}
\text{finv} \ \text{Abs} \ (t \ r) := \{-r, r\}
\text{finv} \ \text{Root} \ (t \ n) \ r := \{\text{if} (0 \leq r) \text{ then } \{r^n\} \text{ else } \varnothing\}
\text{finv} \ \text{Exp} \ (t \ r_0) \ r := \{\text{if} (0 \leq r) \text{ then } \{\exp_0 (r)\} \text{ else } \varnothing\}
\text{finv} \ \text{Log} \ (t \ r_0) \ r := \{r^0\}
\text{finv} \ (\text{Polynomial} \ t \ r_0 \ldots r_m) \ r := \text{polySolve} \ r \ r_0 \ r_1 \ldots r_m
\text{finv} \ (\text{Piecewise} ((t_i e_i)_{i=m}^0) := \text{union} \ ((\text{intersection} \ (\text{finv} \ t_i \ r) \ \text{ (union) } t_{\sum i \leq m} \left(\bigcup \{e_i \vert x\}) \right)_{i=0}^m,
\text{where} \ (x) := \text{vars} t_i
\triangleright \text{Outcomes}

Listing 20. \text{finv} \ computes \ the \ generalized \ inverse \ of \ a \ many-to-one \ transform \ at \ a \ single \ Real.
\[
\text{polyLim} : \text{Real}^* \rightarrow \text{Real}^2
\]
\[
\text{polyLim} \ r_0 \ := \ (r_0, r_0)
\]
\[
\text{polyLim} \ r_0 \ r_1 \ \ldots \ r_m :=
\]
\[
\text{let} \ n \ \text{be} \ \max\{j \mid r_j > 0\}
\]
\[
\text{in} \ \text{if} \ (\text{even} \ n) \ \text{then} \ (\text{if} \ (r_n > 0) \ \text{then} \ \langle \infty, \infty \rangle \ \text{else} \ \langle -\infty, -\infty \rangle)
\]
\[
\text{else} \ (\text{if} \ (r_n > 0) \ \text{then} \ \langle -\infty, \infty \rangle \ \text{else} \ \langle \infty, -\infty \rangle)
\]

**Listing 21.** polyLim computes the limits of a polynomial limits at the infinities.

\[
\text{polySolve} : \text{Real} \rightarrow \text{Real}^* \rightarrow \text{Set}
\]
\[
\text{polySolve} : r \ r_0 \ \ldots \ r_m := \ \text{match} \ r
\]
\[
\begin{aligned}
\triangleright \ (\infty \mid -\infty) & \quad \Rightarrow \ \text{let} \ \langle r_{\text{neg}}, r_{\text{pos}} \rangle & \quad \text{be} \ \text{polyLim} \ r_0 \ \ldots \ r_m \\
\text{in} \ f & \quad \text{be} \ \lambda r'. \ \text{if} \ (r = \infty) \ \text{then} \ (r' = \infty) \ \text{else} \ (r' = -\infty) \\
\text{in} \ \text{let} \ \nu_{\text{neg}} & \quad \text{be} \ \text{if} \ (f \ r_{\text{neg}}) \ \text{then} \ (-\infty) \ \text{else} \ \emptyset \\
\text{in} \ \text{let} \ \nu_{\text{pos}} & \quad \text{be} \ \text{if} \ (f \ r_{\text{pos}}) \ \text{then} \ (\infty) \ \text{else} \ \emptyset \\
\text{in} & \quad \text{if} \ \text{neg} \ \text{union} \ \nu_{\text{pos}} \ \nu_{\text{neg}}
\end{aligned}
\]
\[
\triangleright \ \text{else} \quad \Rightarrow \ \langle \text{roots} \ (r_0 - r) \ r_1 \ \ldots \ r_m \rangle
\]

**Listing 22.** polySolve computes the set of values at which a polynomial is equal to a specific value \(r\).

\[
\text{polyLte} : \text{Boolean} \rightarrow \text{Real} \rightarrow \text{Real}^* \rightarrow \text{Outcomes}
\]
\[
\text{polyLte} \ b \ r \ r_0 \ \ldots \ r_m := \ \text{match} \ r
\]
\[
\begin{aligned}
\triangleright \ -\infty & \quad \Rightarrow \ \text{if} \ b \ \text{then} \ \emptyset \ \text{else} \ \text{polySolve} \ r \ r_0 \ \ldots \ r_m \\
\triangleright \ \infty & \quad \Rightarrow \ \text{if} \ \neg b \ \text{then} \ \langle \text{##} \ -\infty \rangle \ \langle \text{##} \ \infty \rangle
\end{aligned}
\]
\[
\begin{aligned}
\text{else} \ & \quad \text{let} \ \langle \nu_{\text{left}}, \nu_{\text{right}} \rangle \ \text{be} \ \text{polyLim} \ r_0 \ \ldots \ r_m \\
\text{in} \ & \quad \text{let} \ \langle \nu_{\text{left}}, \nu_{\text{right}} \rangle \ \text{be} \ \langle \nu_{\text{left}} = \infty, \nu_{\text{right}} = \infty \rangle \\
\text{in} & \quad \langle \text{b}_{\text{left}} - \infty \rangle \ \langle \text{oo} \text{b}_{\text{right}} \rangle
\end{aligned}
\]
\[
\begin{aligned}
\triangleright \ \text{else} & \quad \Rightarrow \ \text{let} \ \langle r_{\text{k},i} \rangle_{i=1}^{k} \ \text{be} \ \text{roots} \ (r_0 - r) \ r_1 \ \ldots \ r_m \\
\text{in} \ & \quad \text{let} \ \langle \langle r_{\text{left},i}, r_{\text{right},i} \rangle \rangle_{i=0}^{k} \ \text{be} \ \langle \langle -\infty, r_{\text{sk},0} \rangle, \langle r_{\text{sk},1}, r_{\text{sk},2} \rangle, \ldots, \langle r_{\text{sk},k-1}, r_{\text{sk},k} \rangle, \langle r_{\text{sk},k}, \infty \rangle \rangle \\
\text{in} \ & \quad \text{let} \ f_{\text{mid}} \ \text{be} \ \lambda r'. \ \text{if} \ (r = -\infty) \ \text{then} \ r' \\
\text{elseif} & \quad (r' = \infty) \ \text{then} \ r \\
\text{else} & \quad (r + r')/2 \\
\text{t'} & \quad \text{be} \ \text{Poly}(\text{Id}(x)) \ (r_0 - r) \ r_1 \ \ldots \ r_m \\
\text{in} \ & \quad \text{union} \ \text{if} \ T \ [(t')] \ \langle f_{\text{mid}}, r_{\text{left},i}, r_{\text{right},i} \rangle \ \text{then} \ \langle b, r'_{\text{left},i}, r'_{\text{right},i}, b \rangle \\
\text{else} & \quad \emptyset
\end{aligned}
\]

**Listing 23.** polyLte computes the set of values at which a polynomial is less than a given value \(r\).